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RANDOM WATER WAVES ON BEACHES

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SPECTRAL TRANSFORMATION OF RANDOM WATER WAVES ON BEACHES

C. Y. Yang and Y. H. Chen*

1. Introduction

This paper presents the mathematical derivation and physical interpretation of the formula governing the change of energy as random waves propagate on beaches from deep water towards the shoreline. The energy change is characterized by the wave spectral density. The problem is limited to linear, one-dimensional steady waves directed normally to a beach of constant slope. In the wave field it is furthur assumed that there is no wave reflection from the shoreline; no energy input from wind and dissipation from turbulence and no wave-wave interaction.

In five parts the paper first reviews some relevant results of the deterministic wave theory and then examines the classical result of Longuet-Higgins (1) on the transformation of a continuous wave spectrum based on the concept of the conservation of energy flux and the concept of random noise by Rice (2).

To improve on Longuet-Higgins' significant but approximate result where the local slope of the beach is assumed to be horizontal, the more accurate theory by Friedrichs (3) for deterministic waves is used to study the random wave problem.

The basic idea is to determine wave energy changes directly from the exact solution for the wave surface η. This simple approach is possible because the exact solution η for the sloping beach gives the spatial variation in wave amplitude which is not provided by the wave solution for a flat bottom. It is shown that for small beach slopes, the asymptotic solution of Friedrichs leads to the formula governing the transformation of random wave spectrum that is in complete agreement with that of the approximate theory by Longuet-Higgins.

For beach slopes that are not small, a new formula for the wave spectrum

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transformation is derived by extending Stoker's (4) exact solution for deterministic waves to a random wave field. Numerical results based on the new formula are obtained and presented along with those by Longuet-Higgins.

2. Review of Deterministic Water Waves

Consider a one-dimensional, linear, progressive single wave in water of uniform depth h. The wave surface with amplitude a, wave number k, frequency σ, and phase angle θ may be denoted by

$$\eta(x,t) = a \operatorname{Cos}(kx + \sigma t + \theta) \tag{1}$$

The wave energy per unit horizontal surface area is

$$E = \frac{1}{2} \rho g a^2 \tag{2}$$

When such a single wave steadily propagates from deep water, in a normal direction, towards the beach with a uniform beach slope and when it is assumed that there is no energy input from wind and no dissipation due to turbulence within the wave field, then the energy flux must be conserved. That is,

$$EC = E_\infty C_\infty \tag{3}$$

where C is the group velocity and the subscript ∞ denotes the quantity in deep water.

The above results for a single wave can be extended to a multiple of n single waves. Let the combined wave surface be denoted by $\zeta(x,t)$. Then the wave surface and the energy equations are

$$\zeta(x,t) = \sum_n a_n \operatorname{Cos}(k_n x + \sigma_n t + \theta_n) \tag{4}$$

$$E = \frac{1}{2} \rho g \sum_n a_n^2 \tag{5}$$

where the subscript n denotes the nth component wave.

3. Longuet-Higgins' Approximate Theory (1957)

Consider a random progressive wave surface $\eta(x,t)$ denoted by the same equation (1) except that in this case the phase angle θ is a random variable, distributed uniformly in $(0,2\pi)$. Thus $\eta(x,t)$ is a random process. Note that for this random wave model, however, Eqs. (2) and (3) remain valid since the energy and the group velocity for each wave sample are independent of the random phase. All wave samples have equal energy E and group velocity C so that these quantities can be considered as ensemble mean values. The ensemble mean for the random process $\eta(x,t)$ is zero and the variance

$$\mathrm{Var}[\eta(x,t)] = \overline{\eta^2(x,t)} = \frac{1}{2} a^2 \tag{6}$$

where the bar over the symbol denotes ensemble average. Consequently, in view of Eqs. (2) and (6) the variance or the ensemble mean square surface represents the wave energy E except for the constant ρg. This connection leads to the familiar concept of an energy spectrum density. Thus if $S(k)$ is defined by

$$S(k)\Delta k = \mathrm{Var}[\eta(x,t)] \sim E \tag{7}$$

it represents the energy density associated with random waves of wave number k, or simply the wave number spectrum. Using the equation for the conservation of energy flux, Eq. (3), yields the following formula,

$$S(k)\Delta k C = S_\infty(k_\infty)\Delta k_\infty C_\infty \tag{8}$$

If it is assumed that the frequency σ is constant, then the ratio of the group velocities C_∞ to C is equal to that of the wave number increment Δk to Δk_∞

and in this way Longuet-Higgins' important formula may be obtained (1) as

$$S(k) = S_\infty(k_\infty) \tag{9}$$

for the conservation of wave number spectrum. This conservation formula for the single random wave model, Eq. (1), with a single random phase angle, θ, applies to a multiple of n random waves defined by Eqs. (4) and (5), which can be shown as follows.

Let it be assumed that the random phase angles θ_n in Eq. (4) are independently and uniformly distributed in $(0,2\pi)$. Then for the combined wave surface the ensemble mean is zero and the variance

$$\mathrm{Var}[\zeta(x,t)] = \overline{\zeta^2(x,t)} = \sum_n \frac{1}{2} a_n^2 \tag{10}$$

so that the wave energy associated with the component random wave of wave number k_n is $S(k_n)\Delta k$. It is again governed by the relation for the conservation of energy flux, Eq. (8), but under the additional assumption of independent propagation for all component waves. Finally, under the same assumption of constant frequency σ_n for each component wave, the same formula Eq. (9) for the conservation of the wave number spectrum holds.

Thus, it is shown that under the assumption of no wave-wave interaction, the spectrum transformation relation for the random multiple wave model can be derived from a single random wave model. This simplification is used in the following analysis.

The theory is considered approximate because the basic wave surface Eq. (1) and the group velocity C are derived under the assumption of a flat bottom. This assumption is removed in the following.

4. Random Waves on a Shallow Sloping Beach

For a shallow sloping beach, Friedrichs' asymptotic solution (3, p. 114) for a progressive wave is given by, (Figure 1),

$$\eta(x,t) = 2k_\infty^{-1} A(\lambda)\mathrm{Sin}\{\omega^{-1}K(\lambda) + \sigma_\infty t + \frac{\pi}{4}\} \tag{11}$$

where ω is the small beach angle, λ is the ratio of pseudo wave length at x to that at $x \to \infty$ and

$$\omega x k_\infty = \lambda \tanh^{-1} \lambda \tag{12}$$

$$A(\lambda) = \sqrt{\frac{2\pi\omega}{j(\lambda)}} \tag{13}$$

$$j(\lambda) = \lambda + (1 - \lambda^2)\tanh^{-1} \lambda \tag{14}$$

$$K(\lambda) = \tanh^{-1} \lambda + \int_0^\lambda \tanh^{-1} \nu \frac{d\nu}{\nu} \tag{15}$$

In deep water, $\omega x \to \infty$

$$\eta(x,t) = 2k_\infty^{-1}\sqrt{2\pi\omega}\ \mathrm{Sin}(k_\infty x + \sigma_\infty t + \frac{\pi^2}{8\omega} - \frac{\pi}{4}) \tag{16}$$

The term pseudo wave length is used because the wave solution Eq. (11) does not have a wave number term. Consequently only an approximate or pseudo wave length can be defined. Using such a definition for λ in Eq. (12) leads to the well-known dispersion relation for small beach angle, ω.

Since the governing equation for the wave field and the boundary conditions are linear and homogeneous, a phase angle θ can be introduced in Eqs. (11) and (16) to form the following modified solution,

$$\eta(x,t) = 2k_\infty^{-1} A(\lambda)\ \mathrm{Sin}\{\omega^{-1} K(\lambda) + \sigma_\infty t + \frac{\pi}{4} + \theta\} \tag{17}$$

and for $\omega x \to \infty$

$$\eta(x,t) = 2k_\infty^{-1} \sqrt{2\pi\omega}\ \mathrm{Sin}\{k_\infty x + \sigma_\infty t + \frac{\pi^2}{8\omega} - \frac{\pi}{4} - \theta\} \tag{18}$$

Consider now that the phase angle is random with uniform distribution in $(0, 2\pi)$. Then the variance, the spectrum and the transformation for the spectrum can be derived as follows. From Eq. (17) and by definition,

$$\mathrm{Var}[\eta(x,t)] = 4k_\infty^{-2} A^2(\lambda) = S(k)\Delta k \tag{19}$$

From Eq. (18)

$$\mathrm{Var}[\eta(x,t)] = 8k_\infty^{-2}\pi\omega = S_\infty(k_\infty)\Delta k_\infty \tag{20}$$

Dividing Eq. (19) by Eq. (20) yields

$$S(k) = S_\infty(k_\infty) \frac{A^2(\lambda)}{2\pi\omega} \frac{dk_\infty}{dk} \tag{21}$$

Using the definitions of λ, $A(\lambda)$, $j(\lambda)$ and Eq. (12), the above equation is reduced to

$$S(k) = S_\infty(k_\infty) \tag{22}$$

Thus it is shown that based on Friedrichs' asymptotic solution for a shallow sloping beach, the transformation relation for wave number spectrum can be derived directly without the assumption of a flat bottom. The result (Eq. (22)) for shallow slopes agree completely with that from Longuet-Higgins' approximate theory, Eq. (9). The condition of the energy flux conservation and thus the group velocity are not needed in the derivation.

5. Random Waves on Non-Shallow Sloping Beach

For a non-shallow sloping beach of 45°, Stoker's exact solution (4, p. 24) are given by two potential functions, a regular function $\psi_1(x,0,t)$ and a singular function $\psi_2(x,0,t)$,

$$\psi_1(x,0,t) = \frac{\pi}{\sqrt{2}} e^{it} [e^{-x} + \text{Cos } x - \text{Sin } x] \tag{23}$$

$$\psi_2(x,0,t) = \frac{e^{it}}{\sqrt{2}} \{C_i(x) [\text{Sin } x - \text{Cos} x] - [\frac{\pi}{2} + S_i(x)]$$

$$[\text{Cos} x + \text{Sin } x] - e^{-x} E_i(x)\} \tag{24}$$

where for simplicity all quantities in the above two equations and the following equations up to Eq. (30) are dimensionless with x defined as the product of the horizontal dimensional coordinate and the wave number, k_∞; and t as that of the dimensional time and the frequency σ_∞; $S_i(x)$, $C_i(x)$ and $E_i(x)$ are Sine, Cosine, and exponential integral functions, respectively.

From the potential functions two standing wave surface solutions are selected. These are:

$$\eta_1(x,t) = R_e \frac{\partial \psi_1}{\partial t} = -\psi_1(x,0,0) \text{Sin } t \tag{25}$$

$$= - \frac{\pi}{\sqrt{2}} (e^{-x} + \text{Cos } x - \text{Sin } x) \text{Sin } t$$

$$\eta_2(x,t) = I_m \frac{\partial \psi_2}{\partial t} = \psi_2(x,0,0) \text{Cos } t \tag{26}$$

$$= \frac{1}{\sqrt{2}} \{C_i(x) [\text{Sin } x - \text{Cos } x] - [\frac{\pi}{2} + S_i(x)] [\text{Cos } x + \text{Sin } x]$$

$$-e^{-x} E_i(x)\} \text{Cost}$$

These standing waves can be combined as follows, using $\psi_1(x) = \psi_1(x,0,0)$ and $\psi_2(x,0,0) = \psi_2(x)$,

$$\eta(x,t) = -\frac{1}{\pi} [-\psi_1(x) \text{Sint}] - \frac{1}{\pi} [\psi_2(x) \text{Cost}] \tag{27}$$

which, as $x \to \infty$, reduces to a simple progressive wave given by

$$\eta(x,t)_{x \to \infty} = \frac{1}{\sqrt{2}} (\text{Cos } x - \text{Sin } x) \text{Sint} + \frac{1}{\sqrt{2}} (\text{Cos } x + \text{Sin } x) \text{Cost} \tag{28}$$

$$= \text{Sin}(x + t + \frac{\pi}{4})$$

As in the previous section, a phase angle θ is now introduced to obtain the modified wave solutions,

$$\eta(x,t) = \frac{1}{\pi}\,\psi_1(x)\mathrm{Sin}(t+\theta) - \frac{1}{\pi}\,\psi_2(x)\mathrm{Cos}(t+\theta) \tag{29}$$

$$\eta(x,t)_{x\to\infty} = \mathrm{Sin}(x+t+\frac{\pi}{4}+\theta) \tag{30}$$

Having established the wave solutions in proper forms, the phase angle θ is now considered to be a random variable with uniform distribution in $(0, 2\pi)$. Then from Eq. (29) and by definition,

$$\mathrm{Var}[\eta(x,t)] = \frac{1}{2}\,[\frac{1}{\pi^2}\,\psi_1^2(x) + \frac{1}{\pi^2}\,\psi_2^2(x)] = S(k)\Delta k \tag{31}$$

From Eq. (30)

$$\mathrm{Var}[\eta(x,t)_{x\to\infty}] = \frac{1}{2} = S_\infty(k_\infty)\Delta k_\infty \tag{32}$$

Hence $S = S_\infty(k_\infty)\; B(x)\; \frac{dk_\infty}{dk}$ (33)

where $B(x) = \frac{1}{\pi^2}\,[\psi_1^2(x) + \psi_2^2(x)]$ (34)

The transformation relation of wave number spectrum for random water waves on a non-shallow sloping beach of 45° is thus established by Eq. (33), based on Stoker's exact solution. Again, as shown previously, the single random wave solution is applicable to a sum of many random wave components under the assumption of independent phase angles θ_n and independent wave propagation. A comparison of Eq. (33) with Longuet-Higgins' solution, Eq. (9), shows that the correction to the approximate theory is the product $B(x)(dk_\infty/dk)$.

Note that since only the ratio of the variances is relevant in Eqs. (31) and (32), $S(k)$, Δk, $S_\infty(k_\infty)$, Δk_∞ and all quantities to be used hereafter, with the exception of $\psi_1(x)$ and $\psi_2(x)$, can be considered as dimensional for simple physical interpretation.

6. Numerical Results

Since the dispersion relation, $k_\infty = k \tanh kh$ where h is the water depth, is no longer applicable in the exact solution, it is very difficult to calculate accurately the term dk_∞/dk in Eq.(33). Furthermore, in practical applications, the wave frequency spectrum $S(\sigma)$ is generally preferable over the wave number spectrum. Therefore, it is useful to examine the alternate form $S(\sigma)$. Using the frequency spectrum in Eqs. (31) and (32) yields

$$\mathrm{Var}[\eta(x,t)] = \frac{1}{2\pi^2}[\psi_1^2(x) + \psi_2^2(x)] = S(\sigma)\Delta\sigma \tag{35}$$

$$\mathrm{Var}[\eta(\infty,t)] = \frac{1}{2} = S_\infty(\sigma)\Delta\sigma \tag{36}$$

where the same assumption $\sigma = \sigma_\infty$ is used as before. From these equations, the alternate relation for wave energy transformation is obtained as

$$S(\sigma) = \frac{1}{\pi^2}[\psi_1^2(x) + \psi_2^2(x)]\, S_\infty(\sigma) = B(x)S_\infty(\sigma) \tag{37}$$

This form is clearly superior than that with the wave number spectrum $S(k)$, for the troublesome term dk_∞/dk is no longer needed. In terms of $S(\sigma)$, Longuet-Higgins' approximate solution, Eq. (9), becomes

$$S(\sigma) = \frac{C_\infty}{C} S_\infty(\sigma) = \frac{dk}{dk_\infty} S_\infty(\sigma) \tag{38}$$

where the term dk/dk_∞ can be easily calculated according to the dispersion relation of the linear wave theory for a flat bottom, which is consistent with the approximate theory.

The new transformation factor $B(x)$ in Eq. (37) is plotted in Figure 1 together with the factor dk/dk_∞ in Eq. (38) which is equal to C_∞/C and also to E/E_∞ according to the approximate theory.

7. Conclusion

A new formula Eq. (37) is derived for the change of wave energy as random waves from deep water propagate toward the beach. This formula is considered to be an improvement to the well-known one by Longuet-Higgins because the assumption of a flat bottom locally is removed. The basic idea is to use the spatial variation of the wave amplitude from the exact solution of Stoker in the derivation of wave energy rather than the concept of conservation of energy flux.

For a shallow sloping beach, it is shown that the formula derived from Friedrichs' asymptotic solution agrees completely with that by Longuet-Higgins. For a non-shallow sloping beach of 45°, numerical results are computed from the explicit new formula and compared with that by Longuet-Higgins. The comparison shows a general agreement in their spatial variations. The quantitative difference between them is within about 8 percent.

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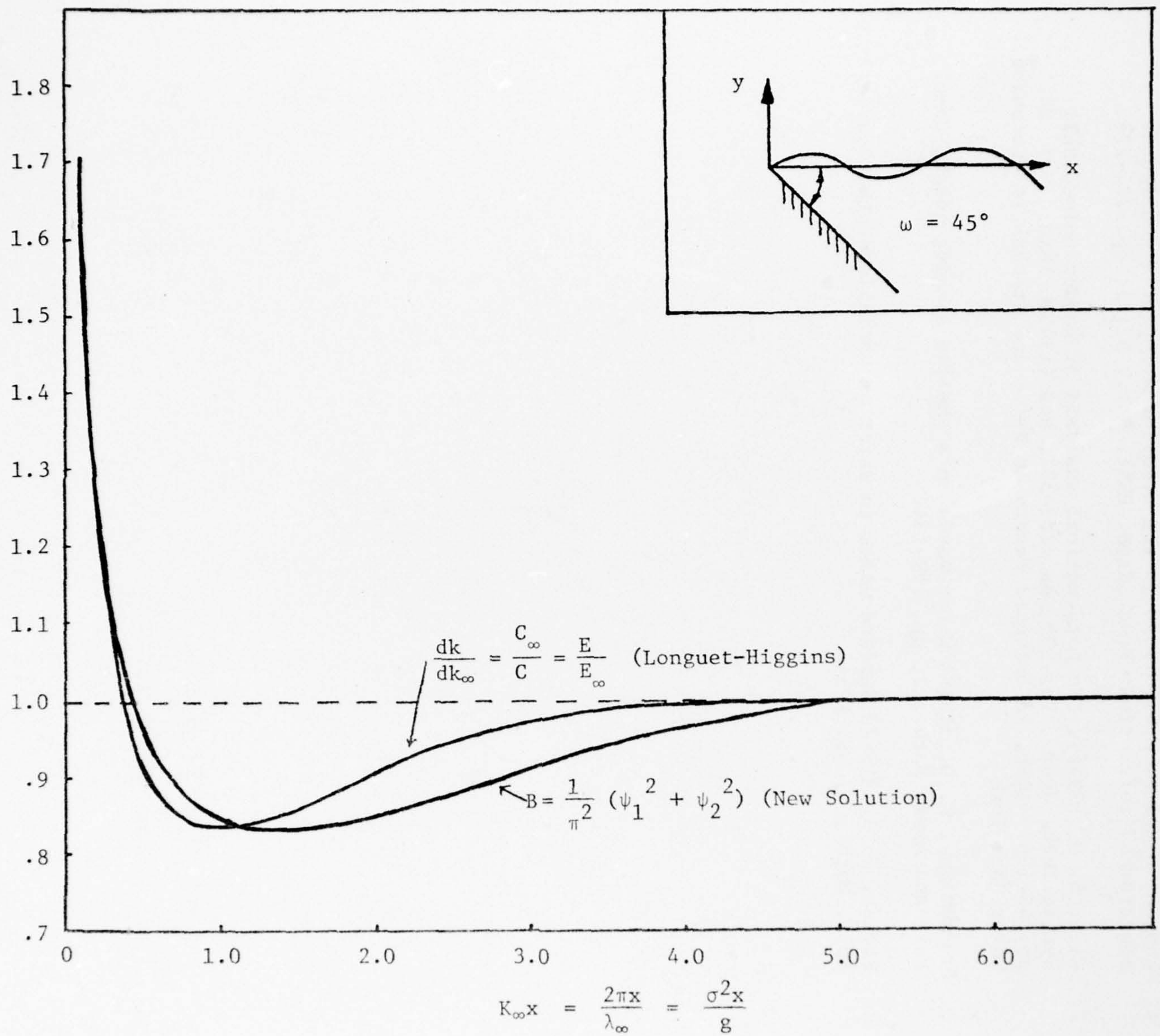

Figure 1 Transfer functions B(x) and dk/dk_∞ for beach of angle 45°.

LINEAR RANDOM WAVES ON WATER OF NONUNIFORM DEPTH

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LINEAR RANDOM WAVES ON WATER OF NONUNIFORM DEPTH

SUMMARY

Propagation of random gravity waves on water of variable depth is examined to incorporate nonuniform bottom effects explicitly into known random wave models assuming a locally flat bottom. The methodology is the same WKB approximation previously employed by Mei, et al. [1968] and Chu and Mei [1970] in the case of monochromatic waves. However, attention is restricted to the linear case and concepts are extended to incoherent random waves which admit an orthogonal spectral representation. This is shown to be valid in a slowly varying inhomogeneous medium just as in a homogeneous case. Nonuniformities in depth introduce a skewness in the free surface whereas, to the same order, the corresponding spectral density remains unchanged. Refractive transformation of continuous spectra is re-examined emphasizing a simplified approach for applications. This is demonstrated with an approximate closed form solution valid for spectral predictions in shallow water. Finally, an inhomogeneous Gaussian wave model is considered. Associated envelope and wave height distributions are derived.

INTRODUCTION

As waves propagate from a homogeneous region such as deep water into a region with uneven underwater topography, their characteristics vary spatially. Prediction of such variations has been systematically investigated in the case of monochromatic waves by several investigators using an approach known as the WKB approximation [see, e.g., Mei, et al. 1968; Chu and Mei, 1970]. The basis of this approach is an asymptotic expansion of the governing equations of wave motion in powers of a small parameter ε which characterizes the slow variation of the still water depth. Formally, ε is proportional to the fractional changes in water depth over a given wavelength for steady state. Partial reflections that always occur at changes in the medium are neglected. However, these are known to be exponentially small. This approach leads to the derivation of a multitude of results on the spatial variation of kinematic-dynamic wave properties in a systematic manner directly through the governing equations of wave motion in contrast with previous wave theories which consider the bottom to be horizontal locally, dealing with the actual depth variation afterwards indirectly through conservation of energy flux.

The motivation here is to apply the WKB approximation to incoherent random waves. To establish the corresponding results in this case, ideas due to Mei, et al. [1968] and Chu and Mei [1970] will be freely drawn on. However, attention is focused to linear waves, and the primary emphasis

is probabilistic with a view to establishing a spatially inhomogeneous random wave model. In the first section, various definitions and formulation of the WKB approximation is summarized, closely following the work of Chu and Mei [1970] with the free surface and the appropriate velocity potential expressed in a continuous spectral form. Subsequently, spatial variations of random amplitudes and free surface spectra, and the explicit effect of an uneven bottom on the free surface and its probability structure are examined. Finally, the probability distribution of wave heights is derived for a Gaussian free surface without any inherent bandwidth constraints on the associated spectral density.

DEFINITIONS AND ANALYSIS

Consider the steady state linear wave propagation-refraction problem, using a coordinate system in which x and y represent the horizontal cartesian coordinates fixed at still water level with the z-axis pointing upwards. Letting $\phi(\underline{x},z,t)$, $\eta(\underline{x},t)$ and $h(\underline{x})$, where $\underline{x} = (x,y)$, denote respectively the velocity potential, the free surface and the still water depth, the problem is governed by

$$\nabla^2\phi + \phi_{zz} = 0 \quad , \quad -h(\underline{x}) \le z \le 0$$

$$g\eta + \phi_t = 0 \quad , \quad z = 0$$

$$\eta_t + \phi_z = 0 \quad , \quad z = 0$$

$$\phi_z + \underline{\nabla}h \cdot \underline{\nabla}\phi = 0 \quad , \quad z = -h(\underline{x}) \qquad (1\ a,b,c,d)$$

In the above, $\underline{\nabla} = (\partial/\partial x,\ \partial/\partial y,\ 0)$ represents the horizontal gradient operator, and $\nabla^2 = \underline{\nabla} \cdot \underline{\nabla}$. It is assumed that the still water depth $h(\underline{x})$ varies slowly over the distance of a given wavelength, λ. This spatial variation is formally characterized by a parameter ε such that

$$\varepsilon = O(|\underline{\nabla} h|/kh) << 1 \tag{2}$$

where $k = 2\pi/\lambda$ is the wave number. In this manner, we can introduce the following slowly varying variables:

$$\underline{\tilde{x}} = (\tilde{x},\tilde{y}) = \varepsilon \underline{x} \quad , \quad \tilde{z} = z \quad , \quad \tilde{t} = \varepsilon t \tag{3}$$

The governing equations can now be rewritten, after combining (1b) and (1c), in the form

$$\varepsilon^2 \tilde{\nabla}^2 \phi + \phi_{\tilde{z}\tilde{z}} = 0 \quad , \quad -h(\underline{\tilde{x}}) \leq \tilde{z} \leq 0$$

$$\phi_{\tilde{t}\tilde{t}} - g\phi_{\tilde{z}} = 0 \quad , \quad \tilde{z} = 0$$

$$\phi_{\tilde{z}} + \varepsilon^2 \underline{\tilde{\nabla}} h \cdot \underline{\tilde{\nabla}} \phi = 0 \quad , \quad \tilde{z} = -h(\underline{\tilde{x}}) \tag{4 a,b,c}$$

where $\underline{\tilde{\nabla}} = (\underline{\nabla}/\varepsilon) = (\partial/\partial\tilde{x},\ \partial/\partial\tilde{y},\ 0)$.

A random wave field which is stationary in time, $\tilde{t}$, and inhomogeneous in the horizontal space, $\underline{\tilde{x}}$, can be represented by

$$\eta(\underline{\tilde{x}},\tilde{t}) = \int_{\underline{k},\omega} dA(\underline{k},\omega,\underline{\tilde{x}}) \exp(i\chi/\varepsilon) \tag{5a}$$

with an associated velocity potential in the form

$$\phi(\tilde{\underline{x}},\tilde{z},\tilde{t}) = \int_{k,\omega} dB(\underline{k},\omega,\tilde{\underline{x}},\tilde{z}) \exp(i\chi/\varepsilon) \tag{5b}$$

The phase function

$$\chi(\tilde{\underline{x}},\tilde{t}) = \int^{\tilde{\underline{x}}} \underline{k} \cdot d\tilde{\underline{x}} - \omega\tilde{t} \tag{6}$$

defines the horizontal vector wave number $\underline{k} = (k_1,k_2)$ and frequency, ω:

$$\underline{k}(\tilde{\underline{x}}) = \tilde{\underline{\nabla}}\chi \quad , \quad \omega = -\chi_{\tilde{t}} \tag{7 a,b}$$

such that

$$\tilde{\underline{\nabla}} x \underline{k} = 0 \quad \text{and} \quad \tilde{\underline{\nabla}}\omega = 0 \tag{8 a,b}$$

It is noted that $A(\underline{k},\omega,\tilde{\underline{x}})$ and $B(\underline{k},\omega,\tilde{\underline{x}},\tilde{z})$ in (5 a,b) are random amplitude processes which are of central importance to the specification of the wave field. For a spatially homogeneous wave field, both A and B become independent of $\tilde{\underline{x}}$ and constitute the usual Fourier-Stieljes representations for the free surface η and the velocity potential ϕ. In particular, the covariance matrices of the increments dA and dB are diagonal entirely as a consequence of the homogeneity condition. This so-called orthogonality property would not strictly be true for a spatially inhomogeneous wave field unless the amplitudes A and B can be regarded as slowly varying functions of $\tilde{\underline{x}}$ such that, corresponding to a gradual variation in depth, h,

the fractional changes in A and B over a given wavelength are much smaller than unity. The orthogonality property is then approximately valid so that

$$E\{dA(\underline{k},\omega,\tilde{\underline{x}})\ dA^*(\underline{k}',\omega',\tilde{\underline{x}})\} = \psi(\underline{k},\omega,\tilde{\underline{x}})\ \delta\underline{k}\ d\omega\ , \quad (\underline{k} = \underline{k}',\ \omega = \omega')$$

$$= \quad 0 \qquad , \qquad \text{otherwise} \tag{9}$$

where $E\{\cdot\}$ denotes a probability or ensemble average for the argument, and $\delta\underline{k} = dk_1 dk_2$ as a shorthand notation. The function ψ is the inhomogeneous spectral density representing locally the density of contributions to the mean square surface deformation, i.e.,

$$E\{|\eta|^2\} = \int_{\underline{k},\omega} \psi(\underline{k},\omega,\tilde{\underline{x}})\ \delta\underline{k}d\omega \tag{10}$$

It is assumed that the random differentials dA and dB admit the following expansions of WKB type [see, e.g., Chu and Mei, 1970]:

$$dA = \sum_{j=0}^{\infty} \varepsilon^j\ dA^{(j)}(\underline{k},\omega,\tilde{\underline{x}}) \quad , \quad dB = \sum_{j=0}^{\infty} \varepsilon^j\ dB^{(j)}(\underline{k},\omega,\tilde{\underline{x}}) \tag{11 a,b}$$

where, for a fixed j, $dA^{(j)}$ and $dB^{(j)}$ are related to one another through (1b) in the form

$$dA^{(j)}(\underline{k},\omega,\tilde{\underline{x}}) = i(\omega/g)\ dB^{(j)}(\underline{k},\omega,\tilde{\underline{x}},0) \tag{12}$$

Substitutions of (11) and (5) into (4 a,b,c) yields a series of perturbation equations for the random differentials $dB^{(j)}$. For j = 0 and 1, to which the analysis will be restricted, we have

$$dB^{(j)}_{\tilde{z}\tilde{z}} - k^2 dB^{(j)} = R^{(j)} \quad , \qquad -h \leq \tilde{z} \leq 0$$

$$g dB^{(j)}_{\tilde{z}} - \omega^2 dB^{(j)} = 0 \quad , \qquad \tilde{z} = 0$$

$$dB^{(j)}_{\tilde{z}} = S^{(j)} \quad , \qquad \tilde{z} = -h \qquad (13\ a,b,c)$$

where

$$R^{(0)} = S^{(0)} = 0$$

$$R^{(1)} = -i \{ 2\underline{k} \cdot \underline{\tilde{\nabla}} dB^{(0)} + dB^{(0)} \underline{\tilde{\nabla}} \cdot \underline{k} \}$$

$$S^{(1)} = -i \underline{\tilde{\nabla}} h \cdot \{ \underline{k}\, dB^{(0)} \}_{\tilde{z} = -h} \qquad (14\ a,b,c)$$

The corresponding solutions to (13a) and (13c) are in the form

$$dB^{(j)}(\underline{k},\omega,\underline{\tilde{x}},\tilde{z}) = dC^{(j)} \cosh Q + \frac{S^{(j)}}{k} \sinh Q$$

$$+ \frac{1}{k^2} \{\sinh Q \int_0^Q R^{(j)} \cosh Q' \, dQ'$$

$$- \cosh Q \int_0^Q R^{(j)} \sinh Q' \, dQ'\} \qquad (15)$$

where $Q = k(\tilde{z} + h)$ and $k = |\underline{k}|$. On substitution from (15), (13b) becomes

$$(k \sinh q - k_\infty \cosh q)\{dC^{(j)} - \frac{1}{k^2}\int_0^q R^{(j)} \sinh Q \, dQ\}$$

$$+ (k \cosh q - k_\infty \sinh q)\{\frac{S^{(j)}}{k} + \frac{1}{k^2}\int_0^q R^{(j)} \cosh Q \, dQ\} = 0 \tag{16}$$

in which $k_\infty = \omega^2/g$ and $q = kh$. In essence, (16) provides the explicit form of the differential coefficient $dC^{(j)}$. For $j = 0$, the system (13 a,b,c) is homogeneous. In this case (1b) leads to the dispersion relation

$$k \tanh q = k_\infty = \omega^2/g, \tag{17}$$

which requires that $dA^{(0)}$ and $dB^{(0)}$ be expressed as

$$dA^{(0)} = dA^{(0)}(\underline{k},\tilde{\underline{x}})$$

$$dB^{(0)} = dC^{(0)} \cosh Q = -i(g/\omega)\, dA^{(0)} \frac{\cosh Q}{\cosh q} \tag{18 a,b}$$

For $j = 1$, and using (14 b,c) and (15), the solution of the inhomogeneous system (13 a,b,c) yields $dB^{(1)}$ in the form

$$dB^{(1)} = dC^{(1)} \cosh Q - i\, dB^{(0)}\, \tilde{F}(\underline{k},\tilde{\underline{x}},\tilde{z}) \tag{19}$$

where

$$\tilde{F}(\underline{k},\tilde{\underline{x}},\tilde{z}) = Q(\tilde{\alpha}_1 + \tilde{\alpha}_2 \tanh Q + \tilde{\alpha}_3 Q) \tag{20}$$

with

$$\tilde{\alpha}_1 = \frac{\underline{k}}{k} \cdot \tilde{\underline{\nabla}} h$$

$$\tilde{\alpha}_2 = \frac{1}{2k} \tilde{\underline{\nabla}} \cdot \left\{ \frac{\underline{k}}{k} \right\} + \frac{\underline{k}}{k^2} \cdot \left\{ \frac{\tilde{\underline{\nabla}} dA^{(0)}}{dA^{(0)}} - \tilde{\underline{\nabla}} q \tanh q \right\}$$

$$\tilde{\alpha}_3 = \frac{\underline{k} \cdot \tilde{\underline{\nabla}} k}{2k^3} \tag{21 a,b,c}$$

In this case, by virtue of the dispersion relation (17), (16) does not provide a unique choice for the differential coefficient $dC^{(1)}$. This coefficient must, therefore, be chosen to give the proper limit in deep water, i.e., as $q \to \infty$, we must have $dB^{(n)} = 0$ for $n \geq 1$. It can be shown then [see, Appendix A] that this requirement leads to the conclusion $dC^{(1)} = 0$, and (19) becomes

$$dB^{(1)} = -i\, dB^{(0)}\, \tilde{F}(\underline{k},\tilde{\underline{x}},\tilde{z}) \tag{22}$$

The preceding solution for $dB^{(1)}$ differs from the corresponding one derived by Chu and Mei [1970]. It is believed here that this is due to an erroneous conlcusion on their part as to the limiting values of the quantities involved in (19) as $q \to \infty$.

With the solution for $dB^{(1)}$ at hand, it is immediate from (12) that

$$dA^{(1)} = -i\, dA^{(0)}\, \tilde{F}(\underline{k},\underline{\tilde{x}},0) \tag{23}$$

and, therefore, up to the order $O(\varepsilon^2)$:

$$\eta(\underline{\tilde{x}},\tilde{t}) = \int_{\underline{k}} [1 - i\varepsilon\, \tilde{F}(\underline{k},\underline{\tilde{x}},0)]\, dA^{(0)} \exp\,(i\chi/\varepsilon)$$

$$\phi(\underline{\tilde{x}},\tilde{z},\tilde{t}) = -ig \int_{\underline{k}} [1 - i\varepsilon\, \tilde{F}(\underline{k},\underline{\tilde{x}},\tilde{z})]\, \frac{dA^{(0)} \cosh Q}{\omega \cosh q} \exp(i\chi/\varepsilon) \tag{24 a,b}$$

It will be expedient from now on to work with the original variables $(\underline{x},z,t)$. Noting that $\underline{\nabla} = \varepsilon\underline{\tilde{\nabla}}$, we define $\alpha_j = \varepsilon\tilde{\alpha}_j$ $(j = 1,2,3)$, $F(\underline{k},\underline{x},z) = \varepsilon\, \tilde{F}(\underline{k},\underline{\tilde{x}},\tilde{z})$, and (24 a,b) become

$$\eta(\underline{x},t) = \int_{\underline{k}} [1 - iF(\underline{k},\underline{x},0)]\, dA^{(0)}(\underline{k},\underline{x}) \exp(i\chi)$$

$$\phi(\underline{x},z,t) = -ig \int_{\underline{k}} [1 - iF(\underline{k},\underline{x},z)]\, \frac{dA^{(0)}(\underline{k},\underline{x}) \cosh Q}{\omega \cosh q} \exp(i\chi) \tag{25 a,b}$$

where

$$\chi(\underline{x},t) = \int^{\underline{x}} \underline{k} \cdot d\underline{x} - \omega t \tag{26}$$

Now, by using a unit vector $\underline{e}_3$ in the z-direction, other relevant properties of the wave field can be summarized up to $O(|\underline{\nabla}h|^2)$ as follows:

<u>Velocity Field</u>

$$\underline{\nabla}\phi + \frac{\partial\phi}{\partial z}\underline{e}_3 = \int_{\underline{k}} \{\underline{\nabla}dB^{(0)} + i\underline{k}[1 - iF(\underline{k},\underline{x},z)]\, dB^{(0)} + dB_z^{(0)}\underline{e}_3\} \exp(i\chi) \tag{27}$$

Acceleration Field

$$\frac{\partial}{\partial t}\left(\underline{\nabla}\phi + \frac{\partial \phi}{\partial z}\underline{e}_3\right) = -i\int_{\underline{k}} \omega\{\underline{\nabla}dB^{(0)} + i\underline{k}[1 - iF(\underline{k},\underline{x},z)]\ dB^{(0)} + dB_z^{(0)}\underline{e}_3\}\exp(i\chi) \tag{28}$$

Free Surface Gradient

$$\underline{\nabla}\eta(\underline{x},t) = \int_{\underline{k}} \{\underline{\nabla}dA^{(0)} + i\underline{k}[1 - iF(\underline{k},\underline{x},0)]\ dA^{(0)}\}\ \exp(i\chi) \tag{29}$$

It is evident in the above results that the differential coefficient $dA^{(0)}$ —and, therefore, $dB^{(0)}$— is so far arbitrary in terms of a probability structure and the explicit dependence on $\underline{x}$. In the following, we will first discuss the spatial dependency of $dA^{(0)}$ and the associated free surface spectra since this can be done in a manner independent of any explicit probabilistic constraints.

SPATIAL DEPENDENCY

The nature of $\underline{x}$-dependency of the random amplitude $dA^{(0)}$ is determined by requiring that $dB^{(0)}$ and $dB^{(1)}$ satisfy the consistency or solvability condition for the inhomogeneous system (13 a,b,c). This condition corresponds to (16), which becomes, on substitution from (14 b,c)

$$\underline{\nabla}h \cdot [\underline{k}\ dB^{(0)}]_{z = -h} + \frac{1}{k}\int_0^q \{2\underline{k}\cdot\underline{\nabla}\ dB^{(0)} + dB^{(0)}\ \underline{\nabla}\cdot\underline{k}\}\cosh Q\ dQ = 0 \tag{30}$$

Making use of (18b) and carrying out the integration, we obtain

$$\frac{\underline{k}}{k} \cdot \frac{\underline{\nabla}\, dA^{(0)}}{dA^{(0)}} = \frac{2\,(q \tanh q - 1)}{\sinh 2q + 2q}\, \frac{\underline{k}}{k} \cdot \underline{\nabla} q - \frac{1}{2}\, \underline{\nabla} \cdot \left\{ \frac{\underline{k}}{k} \right\} \tag{31}$$

In principle, this equation together with the irrotationality condition on $\underline{k}$, i.e. (8a), and boundary conditions associated with $\underline{k}$ and $dA^{(0)}$ provides the description for the spatial variation of $dA^{(0)}$. However, realizing that one is in fact dealing with an ensemble of amplitudes $dA^{(0)}$, a numerical approach which is implicitly required by this description in its present form above is highly undesirable. At the same time, a careful examination of (31) suggests that it can be transformed further by a change of variables from $\underline{x} = (x,y)$ to a curvilinear system (s,n) where s and n denote, respectively, coordinates measured along the orthogonal trajectories (rays) and the crests of a component wavelet as schematically illustrated in Figure 1. This transformation [see, Appendix B] simplifies (31) to a form which can be integrated to give

$$p^{1/2}\, C_g^{1/2}\, dA^{(0)}(\underline{k},\underline{x}) = \text{constant} \tag{32}$$

along a wave ray. In the above, p represents the separation distance between adjacent rays, and C_g is the group speed given in general by

$$C_g = \frac{\partial \omega}{\partial k} = \frac{\omega}{2k} \left\{ \frac{\sinh 2q + 2q}{\sinh 2q} \right\} \tag{33}$$

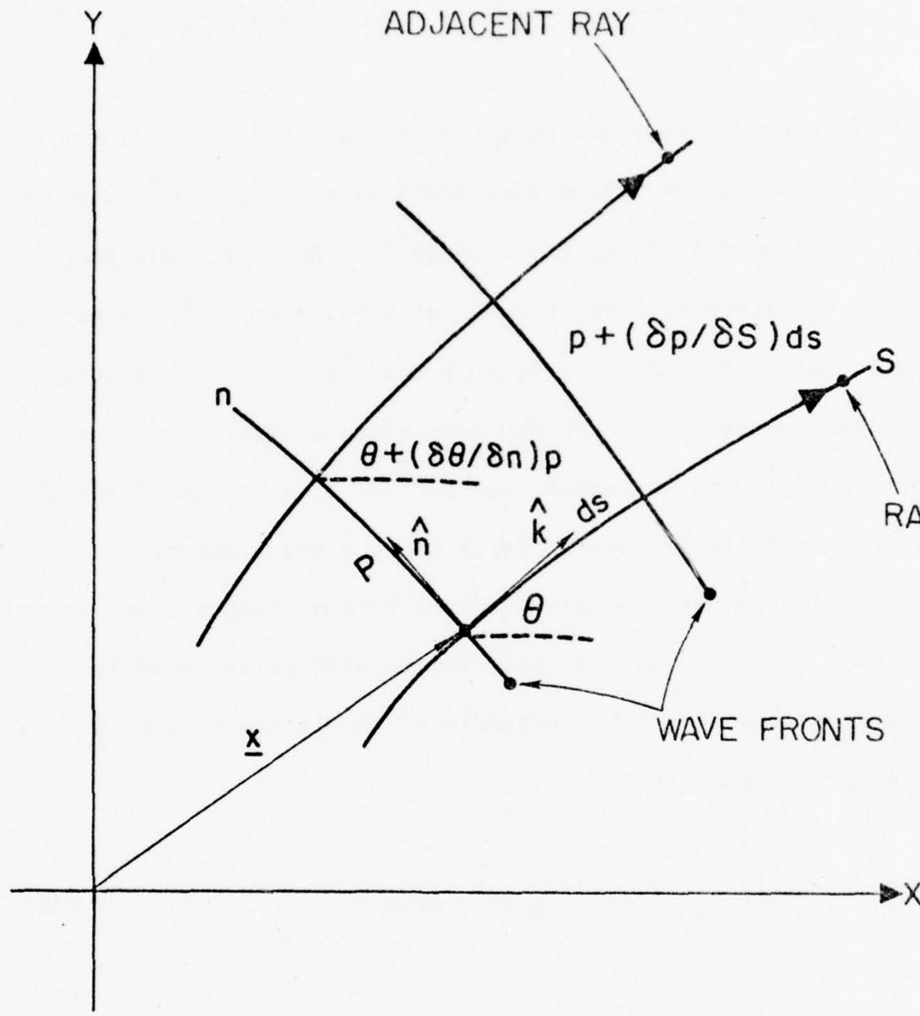

FIGURE 1. Definition Sketch

It is evident that, for waves propagating from a homogeneous region such as deep water into a spatially inhomogeneous regime, (32) implies

$$p_\infty^{1/2}\,(C_g)_\infty^{1/2}\,dA_\infty^{(0)}(\underline{k}_\infty) = p^{1/2}\,C_g^{1/2}\,dA^{(0)}(\underline{k},\underline{x}) \tag{34}$$

where the subscript (∞) designates the constant values in the homogeneous region. It is immediate from (34) that we have the desired relation

$$dA^{(0)}(\underline{k},\underline{x}) = \left\{\frac{p_\infty}{p}\right\}^{1/2}\left\{\frac{(C_g)_\infty}{C_g}\right\}^{1/2} dA_\infty^{(0)}(\underline{k}_\infty) \tag{35}$$

to describe the spatial modulation of the amplitude $dA^{(0)}$. Obviously (35) suggests, first, that $dA^{(0)}$ is linearly proportional to $dA_\infty^{(0)}$. However, since the proportionality constant is real-valued, both $dA^{(0)}$ and $dA_\infty^{(0)}$ have the same phase. Further, since the same constant is also deterministic, the probability structure of $dA^{(0)}$ is readily obtained once that of $dA_\infty^{(0)}$ is specified. Finally, realizing that $A_\infty^{(0)}(\underline{k}_\infty)$ is a random process with orthogonal increments, the most fundamental result embedded in (35) is the proof to the heuristic argument that the inhomogeneous amplitude $A^{(0)}$ and, therefore, $A^{(1)}$, etc., have orthogonal increments if the medium is slowly varying.

At this point, we may note that the ratios

$$\beta = \frac{p}{p_\infty}$$

$$\kappa = \beta^{-1/2} = \left\{\frac{p_\infty}{p}\right\}^{1/2} \tag{36 a,b}$$

are recognized as the ray separation factor and the refraction factor, respectively. The spatial variation of β implicitly required in (35) is governed by the equation of wave intensity, which has been previously discussed in detail by Munk and Arthur [1952], and need not be further elaborated on here. It suffices to point out that the determination of β in general requires numerical calculations, although approximate solutions of the WKB type that may very well be of relevance here are indicated also by the same authors.

INHOMOGENEOUS FREE SURFACE SPECTRA

By virtue of (9), (11a) and (17), the inhomogeneous free surface spectrum can be expressed in the form

$$\psi(\underline{k},\underline{x}) = \psi^{(0)}(\underline{k},\underline{x}) + \psi^{(1)}(\underline{k},\underline{x}) + 0(|\underline{\nabla}h|^2) \tag{37}$$

where

$$\psi^{(0)}(\underline{k},\underline{x}) = E\{|dA^{(0)}|^2\}/\delta\underline{k}$$

$$\psi^{(1)}(\underline{k},\underline{x}) = E\{dA^{(0)}\, dA^{*(1)} + dA^{*(0)}\, dA^{(1)}\}/\delta\underline{k} \qquad (38\ a,b)$$

Now, using (18b), (23) and recognizing that $F(\underline{k},\underline{x},0)$ is real, one can verify that

$$\psi^{(1)}(\underline{k},\underline{x}) = 0 \tag{39}$$

and, (27) becomes

$$\psi(\underline{k},\underline{x}) = \psi^{(0)}(\underline{k},\underline{x}) + 0(|\underline{\nabla}h|^2) \tag{40}$$

Therefore, entirely as a consequence of the fact that the solutions of the two leading orders $dA^{(0)}$ and $dA^{(1)}$ are out of phase by $(\pi/2)$, the free surface spectral density is characterized by $\psi^{(0)}$ corresponding to a locally flat bottom correct up to the order $0(|\underline{\nabla}h|^2)$. Consequently, the spatial transformation of $\psi(\underline{k},\underline{x})$, which is of main interest here, is described to the order $0(|\underline{\nabla}h|^2)$ in terms of the spatial modulation of $dA^{(0)}$. It follows then, by multiplying (32) by its complex conjugate and using (38a),

$$p\, C_g\, \psi^{(0)}(\underline{k},\underline{x})\ \delta\underline{k} = \text{constant} \tag{41}$$

along a wave ray. This is obviously nothing but the conservation of energy flux, and one can further simplify it as previously done by Longuet-Higgins [1956] to obtain the classical result that

$$\psi^{(0)}(\underline{k},\underline{x}) = \psi_\infty^{(0)}(\underline{k}_\infty) = \text{constant} \tag{42}$$

along the wave ray. Another way the same result is obtained is to follow an approach that corresponds in essence to what Chu and Mei [1970] have done in the case of monochromatic waves. In this case, we multiply (30) by $(dA^{*(0)}/\cosh q)$ and use the Leibniz rule to obtain

$$\underline{\nabla} \cdot \int_{-h}^{0} \underline{k}|dB^{(0)}|^2\, dz = 0 \tag{43}$$

Making use of (18b) and ensemble averaging, one can carry out the integration to obtain

$$\underline{\nabla} \cdot [\underline{C}_g \, E\{|dA^{(0)}|^2\}] = 0 \tag{44}$$

where $\underline{C}_g = C_g \, \hat{\underline{k}}$. Now, using (38a) and the fact that $\underline{\nabla} \cdot (\underline{C}_g \, \delta \underline{k}) = 0$ [see, e.g., Phillips, 1969, p. 147], (44) becomes

$$\hat{\underline{k}} \cdot \underline{\nabla} \psi^{(0)} = 0 \tag{45}$$

which is identical to (42), recognizing that $\hat{\underline{k}} \cdot \underline{\nabla} = \partial/\partial s$.

In summarizing the preceding discussion, we observe that the spectral amplitude $\psi^{(0)}$ remains invariant along a wave ray although the associated wave space is distorted due to the spatial modulation of the magnitude and the direction of $\underline{k}$. Consequently, the complete spectral transformation requires, in addition to the amplitude information $\psi^{(0)}$, a knowledge of how the vector wave number $\underline{k}$ varies along a wave ray. This is embedded in the well-known Hamiltonian properties of the wave field [see, e.g., Bretherton and Garrett, 1969], which can be expressed for the steady state case of interest here as

$$\frac{\partial \underline{x}}{\partial s} = \hat{\underline{k}} \quad \text{and} \quad \frac{\partial \underline{k}}{\partial s} = -\frac{1}{C_g} (\underline{\nabla} \omega)_{\underline{k} = \text{const}} \tag{46 a,b}$$

where, from (17),

$$(\underline{\nabla} \omega)_{\underline{k} = \text{const}} = \frac{k\omega}{\sinh 2q} \underline{\nabla} h \tag{47}$$

It is seen that (46 a) obviously describes a wave ray, and (46a) characterizes the variation of $\underline{k}$ along the same ray. In essence, (46 b) represents the kinematical conservation of the vector wave number $\underline{k}$. It is, therefore, a restatement of (8 a,b) since we can show, by recognizing that $\omega = \omega(\underline{k},\underline{x})$, $\underline{k} = \underline{k}(\underline{x})$, and expanding (8 b) with $\underline{\nabla} \times \underline{k} = 0$, that

$$\frac{\partial \omega}{\partial \underline{k}} \cdot \frac{\partial \underline{k}}{\partial \underline{x}} + (\underline{\nabla}\omega)_{\underline{k}} = 0 \tag{48}$$

where

$$\frac{\partial \omega}{\partial \underline{k}} \cdot \frac{\partial \underline{k}}{\partial \underline{x}} = \left[\frac{\partial \omega}{\partial \underline{k}} \cdot \frac{\partial \underline{k}}{\partial x} \quad , \quad \frac{\partial \omega}{\partial \underline{k}} \cdot \frac{\partial \underline{k}}{\partial y}\right]$$

Now, noting that $(\partial\omega/\partial\underline{k}) = C_g\underline{\hat{k}}$ and $\underline{\hat{k}} \cdot (\partial/\partial\underline{x}) = \partial/\partial s$, (46 b) follows.

Consider now random waves propagating from a homogeneous region into a nonuniform-depth regime. The spectral transformation at a point $\underline{x}$ along a ray that originates at $\underline{x}_\infty$ in the homogeneous region where $\underline{k} = \underline{k}_\infty$ and $\psi^{(0)} = \psi_\infty^{(0)}$ is described via (42) and (46 a,b), e.g., on setting s=0 where $\underline{x} = \underline{x}_\infty$ for convenience, we have

$$\psi^{(0)}(\underline{k},\underline{x}) = \psi_\infty^{(0)}(\underline{k}_\infty)$$

$$\underline{x}(s) = \underline{x}_\infty + \int_0^s \underline{\hat{k}}(\underline{x})\, ds$$

$$\underline{k}(\underline{x}) = \underline{k}_\infty - \int_0^s \{(\underline{\nabla}\omega)_{\underline{k}}/C_g\}\, ds \tag{49 a,b,c}$$

The solution to the spatial transformation of $\psi^{(0)}(\underline{k},\underline{x})$ via ray training can involve excessive numerical computation in particular in a situation which requires the determination of $\psi^{(0)}$ for several points in the inhomogeneous region. An alternate approach is to solve (45) together with (8 a) (i.e., $\underline{\nabla \mathrm{x} k} = 0$), using a proper finite difference scheme in the x-plane. The third and possibly best alternative is to take advantage of both of these and use the fact that the transformation of $\psi^{(0)}$ really involves only the spatial modulation of $\underline{k}$-space. Therefore, the whole problem of predicting $\psi^{(0)}(\underline{k},\underline{x})$ reduces to the prediction of a $\underline{k}$-space in the inhomogeneous region corresponding to a specified $\underline{k}_\infty$ in the homogeneous region, i.e., to the solution of $\underline{\nabla \mathrm{x} k} = 0$ subject to the condition $\underline{k} = \underline{k}_\infty$ along the homogeneous boundary. This then enables us to set $\psi^{(0)}(\underline{k},\underline{x}) = \psi^{(0)}(\underline{k}_\infty)$ simply because that particular solution to $\underline{\nabla \mathrm{x} k} = 0$ implies that we can trace a wave ray starting with $\hat{\underline{k}}$ at point $\underline{x}$ in the inhomogeneous region back into the homogeneous boundary where $\underline{k} \to \underline{k}_\infty$. This approach is simplified further by working in a ω,θ-space rather than in $\underline{k}$-space since, in this case, ω is spatially invariant by virtue of (8 b) and, therefore, it would suffice to predict only the spatial modulation of θ in terms of a specified homogeneous boundary value θ_∞. In doing so, it is advantageous to introduce the spectral density $\Omega^{(0)}(\omega,\theta,\underline{x})$ over a ω,θ-plane defined as

$$\Omega^{(0)}(\omega,\theta,\underline{x}) = k\,\frac{\partial k}{\partial \omega}\,\psi^{(0)}(\underline{k},\underline{x}) \tag{50}$$

so that $\Omega^{(0)}\, d\omega d\theta = \psi^{(0)}\delta\underline{k}$. Now, we can easily show, using $\underline{k} = k(\cos\theta,\, \sin\theta)$ in $\underline{\nabla \mathrm{x} k} = 0$, expanding and utilizing the dispersion relation (17) for $(\partial k/\partial x)$

and $(\partial k/\partial y)$ in the resulting expansion, that

$$\underline{C}_g \cdot \underline{\nabla}\theta = -\frac{\omega}{\sinh 2q}\,\hat{\underline{n}} \cdot \underline{\nabla}h \qquad (51)$$

with $\hat{n} = (-\sin\theta, \cos\theta)$, is the required relation governing the spatial modulation of θ. Therefore, (51) with $\theta = \theta_\infty$ being prescribed along the homogeneous boundary and the equation resulting from substitutions of (50) and (33) into (42), i.e.,

$$\Omega^{(0)}(\omega,\theta,\underline{x}) = \frac{k}{k_\infty}\,\frac{(C_g)_\infty}{C_g}\,\Omega_\infty^{(0)}(\omega,\theta_\infty) \qquad (52)$$

are sufficient for the spectral prediction problem. The following example will serve to illustrate this prediction for random waves propagating from deep water into shallow water.

SPECTRAL PREDICTION IN SHALLOW WATER

Consider as an example an underwater topography illustrated in Figure 2. We assume that waves propagate from deep water (i.e., $x = -\infty$) over an intermediate region with parallel isobaths into shallow water where the variation of the still water depth is described (in meters) by

$$h(x,y) = 2\{1 - 10^{-3}\,x^2[1 - e^{-1/2(y/10)^2}]\} \quad ; \quad x > 0\ ,\ |y| < \infty \qquad (53)$$

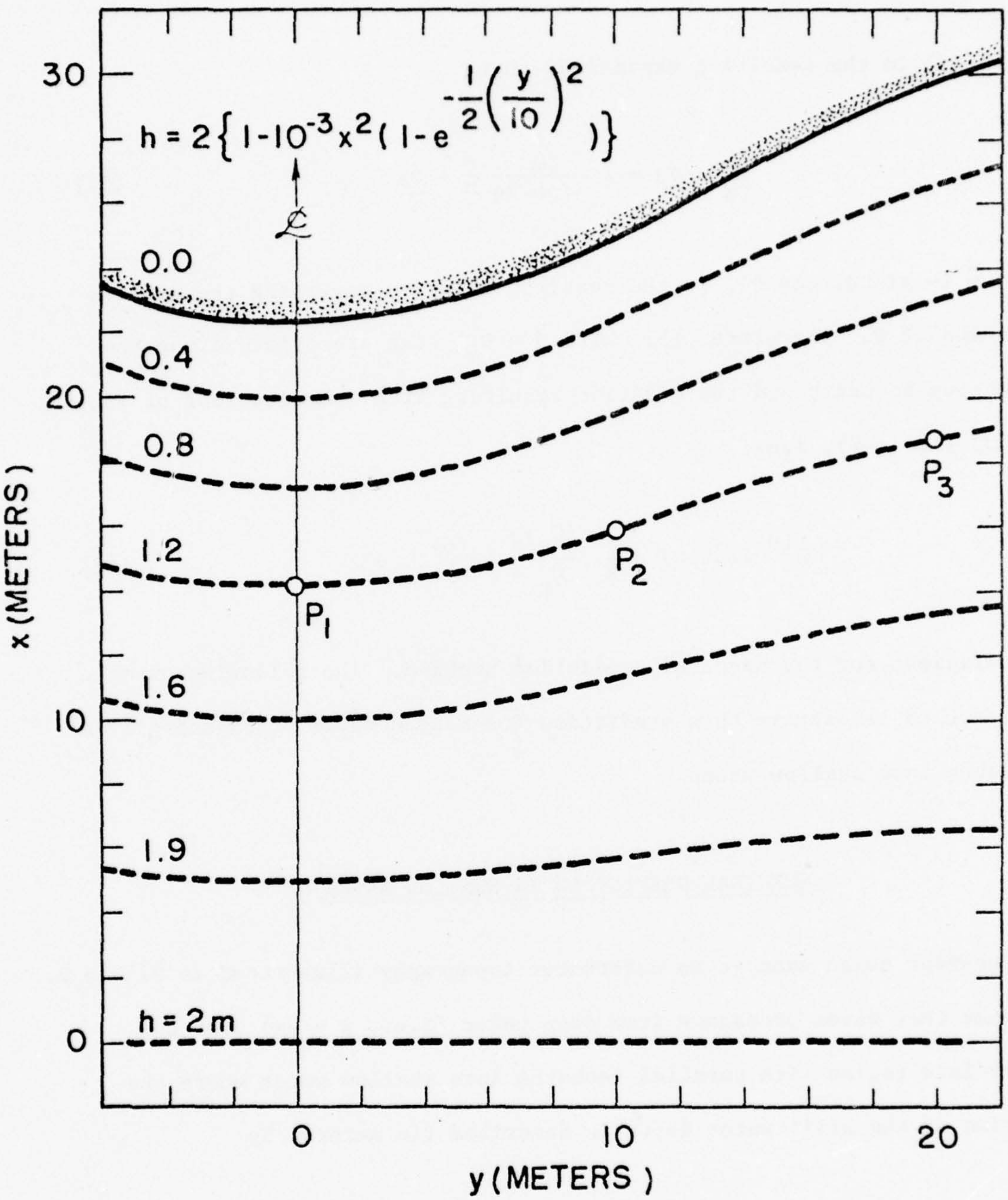

FIGURE 2. Definition Sketch - Spectral Prediction in Shallow Water

The homogeneous deep water wave field is specified in terms of a spectral density of the form

$$\Omega_{\infty}^{(0)}(\omega,\theta_{\infty}) = f_1(\theta_{\infty})\; f_2(\omega) \tag{54}$$

where

$$f_1(\theta_{\infty}) = \frac{8}{3\pi} \cos^4 \theta_{\infty} \quad , \quad |\theta_{\infty}| < \pi/2 \tag{55}$$

represents a directional spreading factor, and

$$f_2(\omega) = (0.0081)\; g^2\omega^{-5} \exp[- 0.74(\frac{g}{U\omega})^4] \quad , \quad \omega > 0 \tag{56}$$

is the well-known Pierson-Moskowitz spectral form, with U representing the overwater wind speed. We take $U = 1.5g$ as a specific case here. The corresponding directional spectral density, $\Omega_{\infty}^{(0)}(\omega,\theta_{\infty})$, is illustrated in Figure 3 showing various contours of $\Omega_{\infty}^{(0)}$ = constant.

In the region from deep water to $x = 0$ where $h = 2m$, the wave field is inhomogeneous only in the x-direction. Therefore, (51) or, more simply, $\underline{\nabla} x \underline{k} = 0$, implies that

$$k \sin \theta = k_{\infty} \sin \theta_{\infty} = \text{const} , \tag{57}$$

which is Snell's law of refraction appropriate to an underwater topography with parallel isobaths. Therefore, it follows from (52) and (57) with

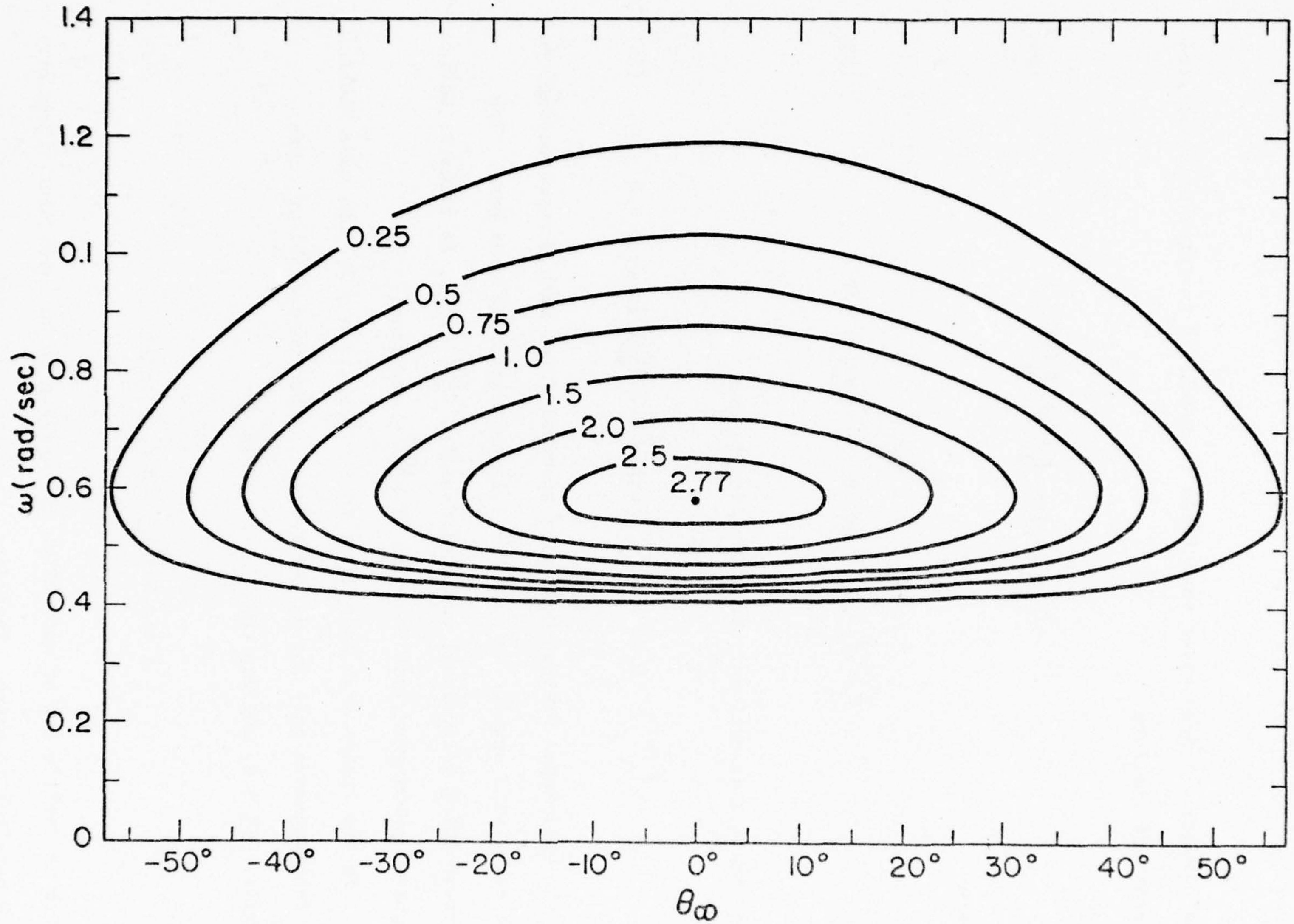

FIGURE 3. Contours of Constant Spectral Density in Deep Water

$(C_g)_\infty = \omega/2k_\infty$ that

$$\Omega^{(0)}(\omega,\theta,\underline{x}) = \frac{k\ \omega}{2\ k_\infty^2\ C_g}\ \Omega_\infty^{(0)}\left[\omega,\ \sin^{-1}\left\{\frac{k}{k_\infty}\ \sin\theta\right\}\right]\ ,\ x < 0,\ |\sin\theta| < 1 \tag{58}$$

For $x \geq 0$, the isobaths are no longer parallel. In this region, the significant spectral components of the wave field, e.g., those for which $.4 \leq \omega \leq 1.2$, are in shallow water, i.e., $(\omega^2 h/g) < (\pi/10)$. Further, it is evident from (57) that propagation directions of these shallow water components are very much focused towards the x-axis. These conditions will enable us to derive an approximate closed form solution for $x > 0$ as follows. First, we expand (57) in the form

$$C_g \cos\theta\ \frac{\partial\theta}{\partial x} + C_g \sin\theta\ \frac{\partial\theta}{\partial y} = \frac{\omega}{\sinh 2q}\left(\sin\theta\ \frac{\partial h}{\partial x} - \cos\theta\ \frac{\partial h}{\partial y}\right) \tag{59}$$

Using small angle approximations, i.e., $\sin\theta \simeq \theta$ and $\cos\theta \simeq 1$, and keeping terms of the first order in θ, (59) becomes

$$C_g\ \frac{\partial\theta}{\partial x} - \frac{\omega}{\sinh 2q}\ \frac{\partial h}{\partial x}\ \theta = -\ \frac{\omega}{\sinh 2q}\ \frac{\partial h}{\partial y} \tag{60}$$

From the definitions of phase speed

$$C = \omega/k = (q\ \tanh q)^{1/2}/k \tag{61}$$

and C_g given by (33), it follows that (60) simplifies to

$$\frac{\partial\theta}{\partial x} - \frac{1}{C}\frac{\partial C}{\partial x}\theta = -\frac{1}{C}\frac{\partial C}{\partial y} \tag{62}$$

or, simply

$$\frac{\partial}{\partial x}\{\theta\, C^{-1}\} = -C^{-2}\frac{\partial C}{\partial y} \tag{63}$$

Integrating, we will obtain

$$\theta(x,y) = C(x,y)\left\{-\int_0^x C^{-2}\frac{\partial C}{\partial y}\,dx + \frac{\theta(0,y)}{C(0,y)}\right\} \tag{64}$$

Since $C = [g\, h(x,y)]^{1/2}$ in shallow water, we can rewrite (64) as

$$\theta(x,y) = \frac{1}{2}\, h^{1/2}(x,y)\left\{-\int_0^x h^{-3/2}\frac{\partial h}{\partial y}\,dx + \frac{2\,\theta(0,y)}{h^{1/2}(0,y)}\right\} \tag{65}$$

For the example of interest here, with h defined by (53), this becomes

$$\theta(x,y) = by\{1 + e^{by^2}\}^{-1}\left\{\left[\frac{h(x,y)}{a(1+e^{-by^2})}\right]^{1/2}\sin^{-1}[1 - \frac{h(x,y)}{2}]^{1/2} - x\right\}$$

$$+ [\frac{h(x,y)}{2}]^{1/2}\,\theta(0,y) \tag{66}$$

where $a = 2 \times 10^{-3}$ and $b = 0.5 \times 10^{-2}$.

Figure 4 illustrates the spectral transformations at three points P_1(14.14m, 0) , P_2(15.78m, 10m) and P_3(18.76m, 20m) of Figure 2, where h = 1.2m. It can be shown from (57) and (66) that

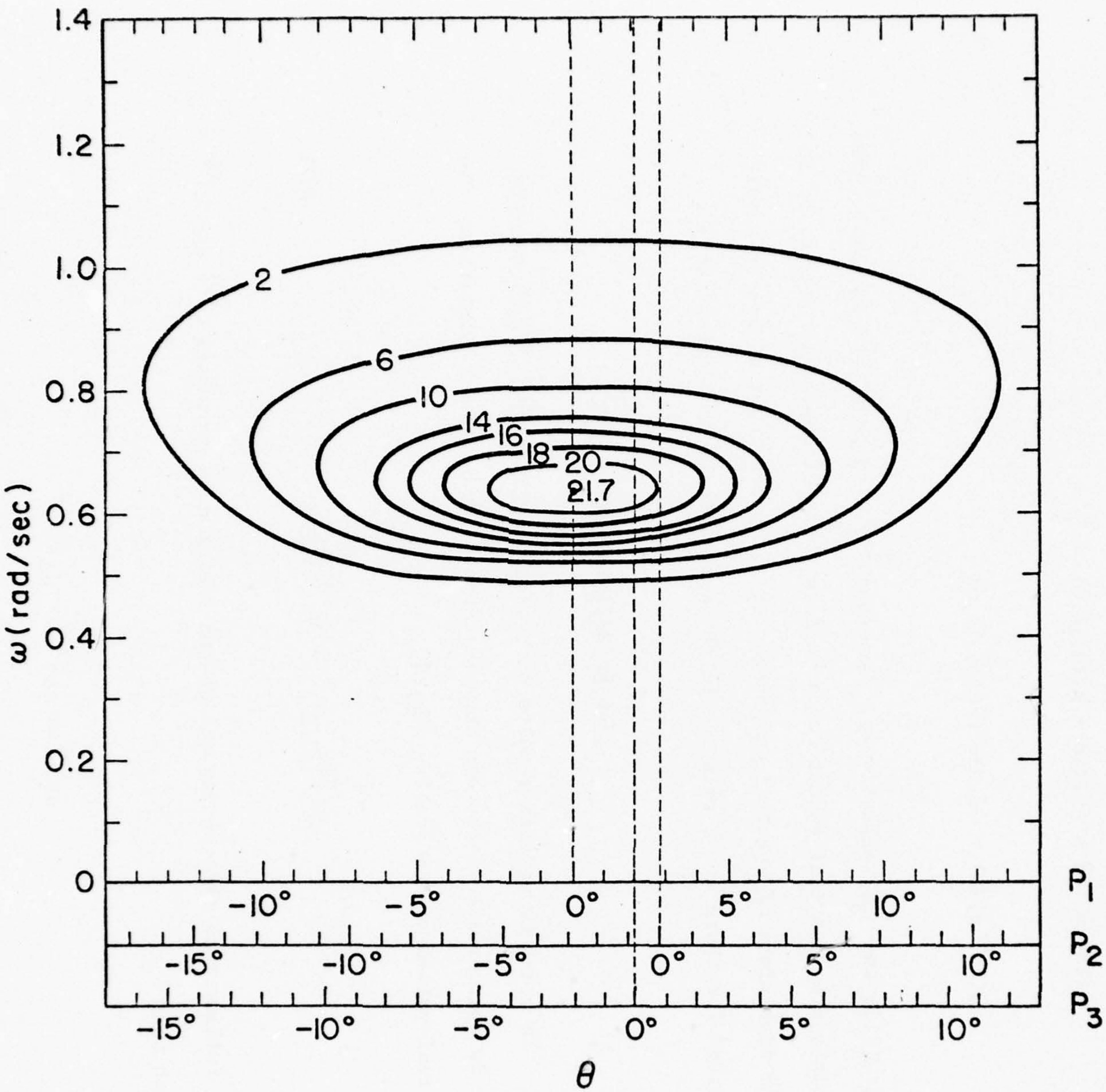

FIGURE 4. Contours of Constant Density at Locations P_1, P_2 and P_3 in Shallow Water

$$\theta(14.14,0) = 0.775\ \theta(0,y) = 0.775 \sin^{-1}\{\frac{k_\infty}{k} \sin\theta_\infty\}$$

$$\theta(15.76,10) = -0.048 + \theta(14.14,0)$$

$$\theta(18.76,20) = -0.036 + \theta(14.14,0)$$

at P_1, P_2 and P_3, respectively. Consequently, the densities at these three points are identical except for a uniform shift of approximately -2.75° in the directions of spectral components at P_2, and a shift of approximately -2.06° for those at P_3 relative to the corresponding components at P_1.

REAL REPRESENTATIONS

In order to utilize results such as those given by (25 a,b) readily, it is appropriate to express them in terms of real-valued quantities. For a real-valued random process $\eta(\underline{x},t)$,

$$\{dA^{(0)}(\underline{k},\underline{x})\}^* = dA^{(0)}(-\underline{k},\underline{x}) \qquad (67)$$

It follows then that we can define two real random processes $U^{(0)}$ and $V^{(0)}$ such that

$$dU^{(0)} = dA^{(0)} + \{dA^{(0)}\}^*$$

$$dV^{(0)} = i[dA^{(0)} - \{dA^{(0)}\}^*] \qquad (68\ a,b)$$

It is evident that $U^{(0)}$ is even whereas $V^{(0)}$ is odd, i.e.,

$$dU^{(0)}(-\underline{k},\underline{x}) = dU^{(0)}(\underline{k},\underline{x})$$

$$dV^{(0)}(-\underline{k},\underline{x}) = -dV^{(0)}(\underline{k}\ \underline{x}) \qquad (69\ a,b)$$

Furthermore, we can show

$$E\{dU^{(0)}\ dV^{(0)}\} = 0$$

$$E\{[dU^{(0)}]^2\} = E\{[dV^{(0)}]^2\} = 2\ E\{|dA^{(0)}|^2\} = 2\ \psi^{(0)}(\underline{k},\underline{x})\ \delta\underline{k} \qquad (70\ a,b)$$

Using (68 a,b), we can rewrite $dA^{(0)}$ in the form

$$dA^{(0)} = \frac{1}{2}\{dU^{(0)} - i\ dV^{(0)}\} \qquad (71)$$

Making use of (71) and also the fact that

$$F(-\underline{k},\underline{x},z) = -\ F(\underline{k},\underline{x},z) \qquad (72)$$

one can transform all the results previously given in a complex form into real representations. In particular, one can verify that the free surface (25 a) becomes to the order $0(|\underline{\nabla}h|^2)$

$$\eta(\underline{x},t) = \frac{1}{2}\int_{\underline{k}}[\{dU^{(0)} - F\ dV^{(0)}\}\cos\chi + \{dV^{(0)} + F\ dU^{(0)}\}\sin\chi] \qquad (73)$$

This can be further simplified if we express $dA^{(0)}$ in terms of its amplitude $|dA^{(0)}|$ and phase μ, i.e.,

$$dA^{(0)} = |dA^{(0)}| \, e^{i\mu} \tag{74}$$

Correspondingly, one would get

$$dU^{(0)} = 2 \, |dA^{(0)}| \cos \mu$$

$$dV^{(0)} = -2 \, |dA^{(0)}| \sin \mu \tag{75 a,b}$$

On substitution from (75 a,b), (73) becomes

$$\eta(\underline{x},t) = \int_{\underline{k}} |dA^{(0)}| \, \{\cos(\chi + \mu) + F \sin(\chi + \mu)\} \tag{76}$$

It should be emphasized that F in the preceding expressions means

$$F = F(\underline{k},\underline{x},0) = q \, (\alpha_1 + \alpha_2 \tanh q + \alpha_3 q) \tag{77}$$

where α_1, α_2, α_3 are as previously defined by (21 a,b,c) with $\tilde{\underline{\nabla}}$ replaced by $\underline{\nabla}$. This function incorporates nonuniform depth effects of the order $O(|\underline{\nabla}h|)$, and constitutes a modification to the locally flat bottom solution represented by the cosine term in (76). However, as previously mentioned, the first two order solutions are orthogonal so that, although the free surface is modulated by the nonuniform depth to the order $O(|\underline{\nabla}h|)$, there is no modification to the free surface spectral density to the same order.

To examine the effect of a sloping bottom on the free surface more explicitly, consider the case of parallel bottom contours such that $\partial h/\partial y = 0$, $\partial h/\partial x \neq 0$. Under these conditions, it can be verified from (21 a,b,c), (31), (57) and (A.4 a,b,c) that (77) simplifies to

$$F(k,x,0) = \frac{q^2\ (3\ \sinh 2q + 2q)}{(\sinh 2q + 2q)^2}\ \frac{dh}{dx} \cos\theta \tag{78}$$

This is illustrated in Figure 5, which shows that the effect of a sloping bottom initially increases at first as waves feel the bottom, then monotonically decreases at shallower depths, both effects being most pronounced for components at normal incidence (i.e., $\theta = 0°$) to the bottom contours as expected. Consider now the net effect of this modulation to the locally flat bottom solution for the free surface regarded as a sum of constituent wavelets of the form

$$\cos(\chi + \mu) + F \sin(\chi + \mu) \tag{79}$$

It is evident that F has the same sign as the local dh/dx. Hence, when $dh/dx < 0$, as we move from a crest (e.g., $\chi + \mu = 0$) of an elementary component $\cos(\chi + \mu)$ to a following trough ($\chi + \mu = \pi$), $F \sin(\chi + \mu) < 0$. On the other hand, as we move up from the same trough up to the next crest, i.e., $\pi < \chi + \mu < 2\pi$, we see that $F \sin(\chi + \mu) > 0$. It follows, therefore, that the profile of an elementary component, $\cos(\chi + \mu)$, of the locally flat bottom solution becomes assymmetrical by a suppression from a crest to the trough followed by a magnification from the trough to the next crest. Clearly, the same profile remains unaffected at any crest or trough since $\sin(\chi + \mu) = 0$ at such points. The situation is exactly reversed when $dh/dx > 0$.

PROBABILITY STRUCTURE AND ENVELOPE PROCESS

The probability structure of a wave field such as that depicted by (76) is embedded in the random character of the complex differential amplitude

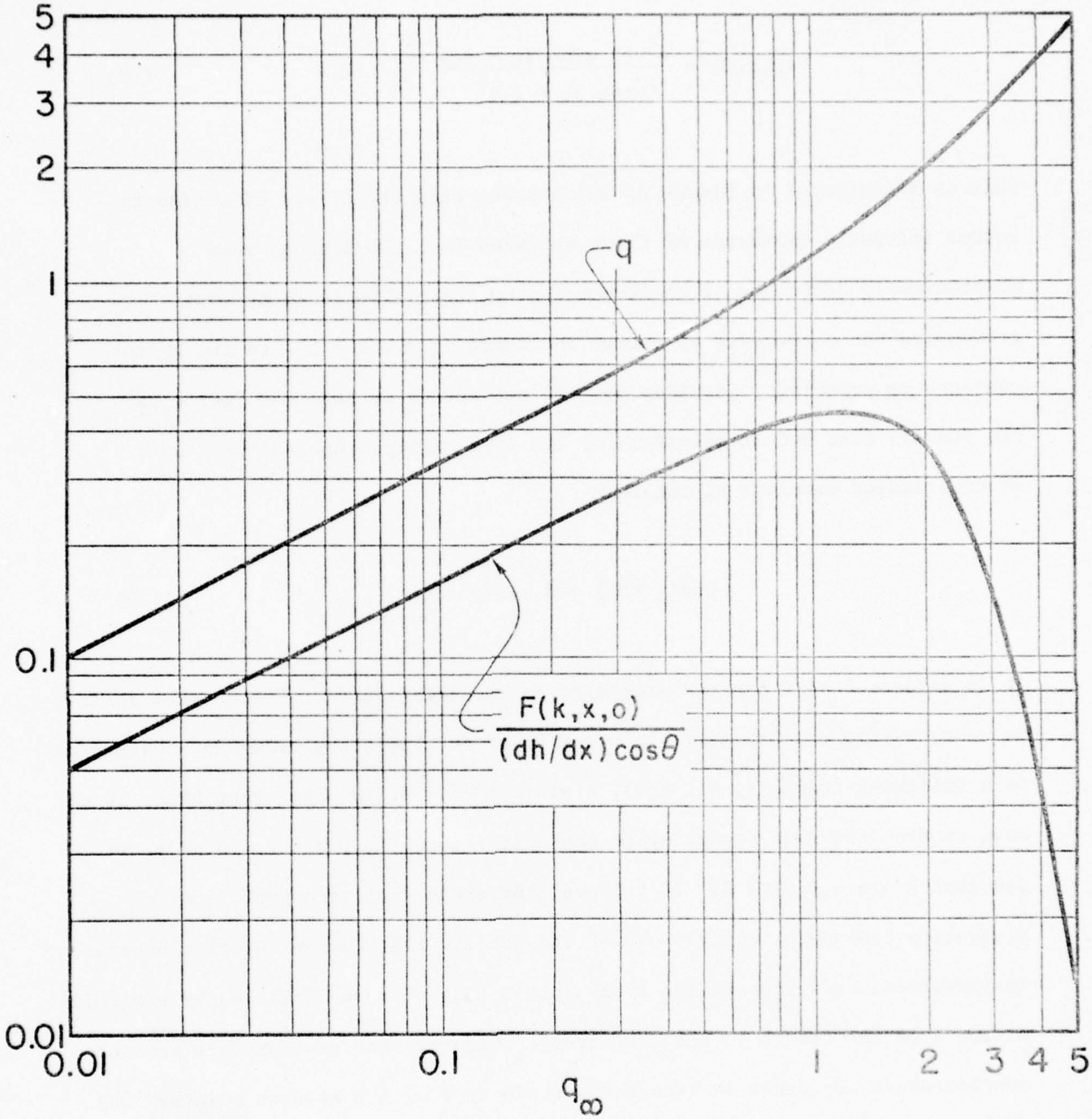

FIGURE 5. Amplitude Modulation of First Order in Bottom Slope

$dA^{(0)}$, in general, in terms of both its amplitude $|dA^{(0)}|$ and phase μ. However, any rational specification as to the explicit forms of these must be guided by the probability structure of $\eta(\underline{x},t)$ itself. In this case it is widely accepted that this structure is Gaussian. This, therefore, leads to a model of random phases for the amplitude process $dA^{(0)}$ which is well-known in theory and applications in the homogeneous cases. The model of random phases simply means -as the name implies- that $|dA^{(0)}|$ is a deterministic function whereas the phases μ are uniformly distributed over $(0, 2\pi)$ and, therefore, independent. In the homogeneous case both $|dA^{(0)}|$ and μ are simply functions of $\underline{k}$ or, equivalently ω and θ, without any dependence on $\underline{x}$. For example, in deep water

$$dA_{\infty}^{(0)}(\underline{k}_{\infty}) = [\psi_{\infty}^{(0)}(\underline{k}_{\infty})\ \delta\underline{k}_{\infty}]^{1/2}\ e^{i\mu(\underline{k}_{\infty})} \tag{80}$$

provides the explicit form required so that

$$E\{|dA_{\infty}^{(0)}|^2\} = \psi^{(0)}(\underline{k}_{\infty})\ \delta\underline{k}_{\infty} \tag{81}$$

An obvious representation for $dA^{(0)}(\underline{k},\underline{x})$ in the spatially inhomogeneous case is in the same form, i.e.,

$$dA^{(0)}(\underline{k},\underline{x}) = [\psi^{(0)}(\underline{k},\underline{x})\delta\underline{k}]^{1/2}\ e^{i\mu(\underline{k},\underline{x})} \tag{82}$$

where, for each $\underline{k}$, $\mu(\underline{k},\underline{x})$ is distributed uniformly over $(0, 2\pi)$. For a wave field which originates from a homogeneous region, (35) implies that

$$\mu(\underline{k},\underline{x}) = \mu(\underline{k}_\infty) \tag{83}$$

as a general invariant property. In this manner one can obtain from (76), on substitution from (82), a spatially inhomogeneous random wave field which is Gaussian in probability structure [see, Appendix C], i.e.,

$$f_\eta(u;\underline{x})\,du = \text{Prob}\{u \leq \eta(\underline{x},t) \leq u + du\}$$

$$= \frac{1}{\sigma_\eta \sqrt{\pi}} \exp\{-\frac{1}{2}(\frac{u}{\sigma_\eta})^2\}\,du \tag{84}$$

where

$$\sigma_\eta{}^2 = \sigma_\eta{}^2(\underline{x}) = E\{\eta^2(\underline{x},t)\} \tag{85}$$

One of the useful descriptions associated with a random wave surface of the form (25 a) is its envelope process which is defined, following Cramer and Leadbetter [1967], at a fixed $\underline{x}_o$ as

$$\Xi(t) = [\eta^2(\underline{x}_o,t) + \hat{\eta}^2(\underline{x}_o,t)]^{1/2} \tag{86}$$

where $\Xi(t) \geq |\eta(\underline{x}_o,t)|$ and $\Xi(t) = |\eta(\underline{x}_o,t)|$ when $\hat{\eta}(\underline{x}_o,t) = 0$. It is known [see, e.g., Crandall, 1970; Yang, 1972] that the envelope definition of (86) is essentially the same as that due to Rice [1955] for narrow-band processes. In (86), $\hat{\eta}(\underline{x}_o,t)$, which is called the Hilbert transform of $\hat{\eta}(\underline{x}_o,t)$, is obtained by passing $\eta(\underline{x}_o,t)$ through a filter with the frequency response

$$G(\omega) = \begin{cases} i \ , & \omega < 0 \\ 0 \ , & \omega = 0 \\ -i \ , & \omega > 0 \end{cases} \tag{87}$$

or, simply $G(\omega) = -i\ \mathrm{sgn}(\omega)$, where $\mathrm{sgn}(\omega)$ designates the sign of ω. Therefore, it follows from (87) and (25 a) that

$$\hat{\eta}(\underline{x}_o,t) = -i \int_{\underline{k}} \mathrm{sgn}(\omega)\ [1 - iF(\underline{k},\underline{x}_o,0)]\ dA^{(0)}(\underline{k},\underline{x}_o)\ \exp(i\chi) \tag{88}$$

Now, by making use of (67) through (72), and associating $-\underline{k}$ with $-\omega$ and $+\underline{k}$ with $+\omega$, we can rewrite (88) in the following equivalent real form:

$$\hat{\eta}(\underline{x}_o,t) = \int_{\underline{k}} \mathrm{sgn}(\omega)\ |dA^{(0)}|\{\sin(\chi + \mu) + F\cos(\chi + \mu)\} \tag{89}$$

Assume now that $\eta(\underline{x}_o,t)$ is Gaussian. It follows then that $\hat{\eta}(\underline{x},t)$ is also Gaussian. Furthermore, we can verify, from (76) and (89) with μ being uniformly distributed in $(0, 2\pi)$, that

$$\sigma_{\hat{\eta}}^{\,2}(\underline{x}_o) = \sigma_{\eta}^{\,2}(\underline{x}_o) = E\{\eta^2(\underline{x}_o,t)\}$$

$$E\{\eta(\underline{x}_o,t)\ \eta(\underline{x}_o,t+\tau)\} = E\{\hat{\eta}(\underline{x}_o,t)\ \hat{\eta}(\underline{x}_o,t+\tau)\}$$

$$= \frac{1}{2}\int_{\underline{k}} \psi^{(0)}(\underline{k},\underline{x}_o)\ \{\cos\omega\tau + F\sin\omega\tau\}\ \delta\underline{k}$$

$$E\{\eta(\underline{x}_o,t)\ \hat{\eta}(\underline{x}_o,t)\} = \int_{\underline{k}} \mathrm{sgn}(\omega)\ F(\underline{k},\underline{x}_o,0)\ \psi^{(0)}(\underline{k},\underline{x}_o)\ \delta\underline{k} \tag{90 a,b,c}$$

Letting,

$$\rho(\underline{x}_o) = \frac{E\{\eta(\underline{x}_o,t)\ \hat{\eta}(\underline{x}_o,t)\}}{\sigma_\eta^{\ 2}} \tag{91}$$

the joint probability density of $\eta(\underline{x}_o,t)$ and $\hat{\eta}(\underline{x}_o,t)$ becomes

$$f_{\eta\hat{\eta}}(\eta,\hat{\eta}\ ;\ \underline{x}_o) = \frac{1}{2\pi\ \sigma_\eta^{\ 2}(1-\rho^2)^{1/2}} \exp\left\{-\frac{\eta^2 - 2\rho\ \eta\hat{\eta} + \hat{\eta}^2}{2\ \sigma_\eta^{\ 2}(1-\rho^2)}\right\}\ ,\ -\infty < \eta,\hat{\eta} < \infty \tag{92}$$

Now, we can use the fact

$$\text{Prob}\{\ \Xi(t) \leq u\} = \text{Prob}\{[\eta^2 + \hat{\eta}^2]^{1/2} \leq u\}$$

to show that the density of $\Xi(t)$ is

$$f_\Xi\ (u;\underline{x}_o) = \frac{u}{2\pi\ \sigma_\eta^{\ 2}(1-\rho^2)^{1/2}}\ \exp\left\{-\frac{u^2}{2\ \sigma_\eta^{\ 2}(1-\rho^2)}\right\} \cdots$$

$$\cdots\int_o^{2\pi} \exp\left\{\frac{\rho\ u^2\ \sin\ 2v}{2\ \sigma_\eta^{\ 2}(1-\rho^2)}\right\} dv \tag{93}$$

Recognizing further that

$$\int_0^{2\pi} \exp\left\{ \frac{\rho\, u^2 \sin 2v}{2\,\sigma_\eta^{\,2}(1-\rho^2)} \right\} dv = 2\pi\, I_o\left[\frac{\rho\, u^2}{2\,\sigma_\eta^{\,2}(1-\rho^2)}\right] \tag{94}$$

where $I_o[\cdot]$ represents the zero-order modified Bessel function of the first kind, (93) becomes

$$f_\Xi\,(u;\underline{x}_o) = \frac{u}{\sigma_\eta^{\,2}(1-\rho^2)^{1/2}} \exp\left\{- \frac{u^2}{2\,\sigma_\eta^{\,2}(1-\rho^2)}\right\} I_o\left[\frac{\rho\, u^2}{2\,\sigma_\eta^{\,2}(1-\rho^2)}\right] ,$$

$$0 \leq u < \infty \tag{95}$$

Finally, it can be verified that the j-th moment of the envelope process is in the form

$$E\{\,\Xi^j\} = \pi^{-1/2}(2\sigma_\eta)^j(1-\rho^2)^{\frac{j+1}{2}}\quad \Gamma(\tfrac{j}{4}+1)\;\Gamma(\tfrac{j}{4}+\tfrac{1}{2})\,\mathcal{F}(\tfrac{j}{4}+1,\;\tfrac{j}{4}+\tfrac{1}{2};\;1;\;\rho^2) \tag{96}$$

where Γ and $\mathcal{F}$ are the gamma function and the hypergoemetric function, respectively. The distribution of wave heights, $H = 2\Xi$, follows immediately from (95), i.e.,

$$f_H(v;\underline{x}_o) = \frac{1}{2}\, f_\Xi\,(\tfrac{v}{2};\underline{x}_o) \tag{97}$$

Hence, by recognizing that $E\{H^2\} = \overline{H^2} = 4\,\sigma_\eta^{\,2}$, we obtain

$$f_H(v;\underline{x}_o) = \frac{v}{\overline{H^2}(1-\rho^2)^{1/2}} \exp\left\{-\frac{v^2}{2\overline{H^2}(1-\rho^2)}\right\} I_o\left[\frac{\rho v^2}{2\overline{H^2}(1-\rho^2)}\right] ,$$

$$0 \leq v < \infty \tag{98}$$

The only constraint on the preceding results is the assumption that the free surface $\eta(\underline{x},t)$ is Gaussian. In deep water or in a region of uniform depth where $F \to 0$, $\rho = 0$ and, therefore, (98) and (99) become the Rayleigh density and the associated j-th moment, respectively, i.e.,

$$f_H(v;\underline{x}_o) = \frac{v}{\overline{H^2}} \exp\left\{-\frac{v^2}{2\overline{H^2}}\right\} , \quad 0 \leq v < \infty$$

$$E\{H^j\} = (2\,\overline{H^2})^{j/2}\,\Gamma(\tfrac{j}{2}+1) \tag{100 a,b}$$

These results, which are widely used in theory and applications with the inherent constraint that the underlying free surface spectrum is narrow-band, are in fact valid without any bandwidth restrictions. It is, therefore, not surprising that most empirical wave height frequency curves obtained in field applications involving wide-band spectra support this conclusion. Nonuniform depth effects on the distribution of wave heights are characterized to the order $O(|\nabla h|^2)$ by ρ, which is simply the correlation coefficient of $\eta(\underline{x}_o,t)$ and $\hat{\eta}(\underline{x}_o,t)$. This correlation introduces a skewness in the Rayleigh form appropriate to a flat bottom towards lower wave height values. For example, it can be verified from (99) and (100 b) with $j = 1$ that the ratio between the mean values of H corresponding to a variable bottom

and a flat one is $(1 - \rho^2)\mathcal{F}(5/4;\ 3/4;\ 1;\ \rho^2)$, which is always less than unity since $\rho^2 < 1$.

CONCLUDING REMARKS

Propagation and refraction of linear waves in water of nonuniform depth have been extended to the case of random waves characterized by a two-dimensional spectral distribution. Attention was focused to the orthogonal representation of the free surface, its spectral density and to the statistical distribution of wave heights with the primary objective to incorporate nonuniform depth effects explicitly into well-known solutions associated with a locally flat bottom. Although the methodology employed corresponds exactly to the linear approximation of that developed by Chu and Mei [1970] for the case of monochromatic waves, the solution derived here for the order $O(|\underline{\nabla} h|)$ is different, and believed to be the valid one. What is, however, more important is the extension of the results to a random wave field and to a continuous spectral representation. This, therefore, compliments and generalizes the deterministic concepts, leading to various probabilistic descriptions of wave properties in spatially inhomogeneous media. The probabilistic description was also limited to the free surface and its envelope. The extension of these to the surface gradient and other kinematic and dynamic properties such as pressure, velocities, etc. does not offer any special difficulty and needs to be done. In particular, the predictions of wave breaking and breaker line for waves impinging on a beach can be formulated as a smooth generalization to what Biesel [1952] and Gaughan and Komar [1975] have done in the case of one-dimensional monochromatic waves. It must, however, be pointed out that a definite

discrepancy exists between the solution of the order $O(|\underline{\nabla}h|)$ here and the corresponding one used by these authors. These and various other aspects of the problem such as an extension to include nonlinearities are considered for future study.

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APPENDIX A - Derivation of Coefficient $dC^{(1)}$

To determine $dC^{(1)}$, we use the condition that as $q \to \infty$, $dB^{(1)} = 0$.

Therefore, from (19) and (18 b),

$$dC^{(1)} = \lim_{q \to \infty} \left\{ \frac{(g/\omega)\, dA^{(0)}\, \tilde{F}(\underline{k}, \tilde{\underline{x}}, z)}{\cosh q} \right\} \tag{A.1}$$

or, on substitution from (20) and (21 a,b,c),

$$(\omega/g)\, dC^{(1)} = \lim_{q \to \infty} dA^{(0)} \{ \tilde{\alpha} \frac{Q}{\cosh q} + \tilde{\alpha}_2 \frac{Q \tanh Q}{\cosh q} + \tilde{\alpha}_3 \frac{Q^2}{\cosh q} \} \tag{A.2}$$

We can now verify, by using (14 a,b) and (18 b) in (16) and integrating, that

$$\frac{\underline{k}}{k} \cdot \frac{\tilde{\underline{\nabla}}\, dA^{(0)}}{dA^{(0)}} = \frac{\underline{k}}{k} \cdot \{ \frac{2\, \tilde{\underline{\nabla}}\, q\, (q \tanh q - 1)}{\sinh 2q + 2q} + \frac{\tilde{\underline{\nabla}} k}{2k} \} - \frac{\tilde{\underline{\nabla}} \cdot \underline{k}}{2k^2} \tag{A.3}$$

where, from (17),

$$\tilde{\underline{\nabla}} k = - \frac{2k^2\, \tilde{\underline{\nabla}} h}{\sinh 2q + 2q}$$

$$\tilde{\underline{\nabla}} \cdot \underline{k} = - \frac{2k^3}{\sinh 2q + 2q} \left(\frac{1}{k_1} \frac{\partial h}{\partial \tilde{x}} + \frac{1}{k_2} \frac{\partial h}{\partial \tilde{y}} \right)$$

$$\tilde{\underline{\nabla}} q = k\, \tilde{\underline{\nabla}} h \left(\frac{\sinh 2q}{\sinh 2q + 2q} \right) \tag{A.4 a,b,c}$$

On the basis of the preceding and using the definitions of $\tilde{\alpha}_1$, $\tilde{\alpha}_2$ and $\tilde{\alpha}_3$ given by (21 a,b,c) it immediately follows that $dC^{(1)} = 0$.

APPENDIX B - Coordinate Transformation and Derivation of Equation (32)

Consider a change of variables from $\underline{x} = (x,y)$ to an orthogonal curvilinear system (s,n), where s and n denote coordinates measured along and perpendicular to the orthogonal trajectories (rays) of a component wavelet, as schematically illustrated in Figure 1. Referring to the same figure, we see that

$$\hat{\underline{k}} = \frac{\underline{k}}{k} = (\cos\theta,\ \sin\theta)$$

$$\hat{\underline{n}} = (-\sin\theta,\ \cos\theta) \qquad \text{(B.1 a,b)}$$

represent unit vectors in the direction of s and n at $\underline{x}$. Furthermore, designating the separation distance between two neighboring s trajectories by p, we have

$$\frac{\partial\theta}{\partial n} = \frac{1}{p}\frac{\partial p}{\partial s} \qquad \text{(B.2)}$$

Now, for an arbitrary function $f = f(x,y)$, we can write

$$\frac{\partial f}{\partial s} = \hat{\underline{k}} \cdot \underline{\nabla} f = \cos\theta \frac{\partial f}{\partial x} + \sin\theta \frac{\partial f}{\partial y}$$

$$\frac{\partial f}{\partial n} = \hat{\underline{n}} \cdot \underline{\nabla} f = -\sin\theta \frac{\partial f}{\partial x} + \cos\theta \frac{\partial f}{\partial y} \qquad \text{(B.3 a,b)}$$

and, on solving for $\partial f/\partial x$ and $\partial f/\partial y$

$$\frac{\partial f}{\partial x} = \cos\theta \frac{\partial f}{\partial s} - \sin\theta \frac{\partial f}{\partial n}$$

$$\frac{\partial f}{\partial y} = \sin\theta \frac{\partial f}{\partial s} + \cos\theta \frac{\partial f}{\partial n} \qquad \text{(B.4 a,b)}$$

We can then use (B.4 a,b) for cos θ and sin θ, respectively, to obtain

$$\frac{\partial}{\partial x}(\cos\theta) = \cos\theta \frac{\partial}{\partial s}(\cos\theta) - \sin\theta \frac{\partial}{\partial n}(\cos\theta)$$

$$\frac{\partial}{\partial y}(\sin\theta) = \sin\theta \frac{\partial}{\partial s}(\sin\theta) + \cos\theta \frac{\partial}{\partial n}(\sin\theta) \qquad \text{(B.5 a,b)}$$

Combining (B.5 a) and (B.5 b), and utilizing (B.2) yields

$$\underline{\nabla} \cdot \hat{\underline{k}} = \underline{\nabla} \cdot \left\{\frac{\underline{k}}{k}\right\} = \frac{1}{p}\frac{\partial p}{\partial s} \qquad \text{(B.6)}$$

The group velocity is defined by

$$\underline{C_g} = C_g\,\hat{\underline{k}} = \frac{\omega}{2k}\left(\frac{\sinh 2q + 2q}{\sinh 2q}\right)\hat{\underline{k}} \qquad \text{(B.7)}$$

In deep water as $q \to \infty$, this becomes a constant independent of $\underline{x}$, i.e.,

$$(\underline{C}_g)_\infty = (C_g)_\infty \; \hat{\underline{k}}_\infty = \frac{\omega}{2k} \; \hat{\underline{k}}_\infty \tag{B.8}$$

Now, using the definitions (B.7) and (B.8), we can show that

$$\frac{1}{2} \frac{\partial}{\partial q} \; \{\ln \frac{(C_g)_\infty}{C_g}\} = \frac{2(q \tanh q - 1)}{\sinh 2q + 2q} \tag{B.9}$$

On the basis of the preceding results and definitions, (31) becomes

$$\frac{1}{dA^{(0)}} \frac{\partial}{\partial s} \{dA^{(0)}\} = \frac{1}{2} \frac{\partial}{\partial q} \{\ln \frac{(C_g)_\infty}{C_g}\} \frac{\partial q}{\partial s} - \frac{1}{2p} \frac{\partial p}{\partial s} \tag{B.10}$$

or, simply

$$\frac{1}{dA^{(0)}} \; \frac{\partial}{\partial s} \{dA^{(0)}\} = \frac{1}{2} \frac{\partial}{\partial s} \; \{\ln \frac{(C_g)_\infty}{C_g}\} \; - \frac{1}{2p} \frac{\partial p}{\partial s} \tag{B.11}$$

Realizing that $(C_g)_\infty$ = constant for a fixed $\underline{k}$, and integrating (B.11), we obtain the desired result that

$$p^{1/2} \; C_g^{\;1/2} \; dA^{(0)}(\underline{k},\underline{x}) = \text{constant} \tag{B.12}$$

along a wave ray.

APPENDIX C - Proof of Gaussian Property

A mean-zero random process is Gaussian if its log-characteristic function possesses the following form:

$$\ln M_{\eta}\ \{\tau(\underline{x},t)\} = -\frac{1}{2}\int_{\underline{X}}\int_{\underline{X}}\int_{T}\int_{T} K_{\eta\eta}(\underline{x}_1,\ \underline{x}_2,\ t_1,\ t_2)\ \tau(\underline{x}_1,t_1)\ \ldots$$

$$\ldots\ \tau(\underline{x}_2,\ t_2)\ d\underline{x}_1\ d\underline{x}_2\ dt_1\ dt_2 \tag{C.1}$$

where

$$K_{\eta\eta}(\underline{x}_1,\ \underline{x}_2,\ t_1,\ t_2) = E\{\eta(\underline{x}_1,\ t_1)\ \eta^*(\underline{x}_2,\ t_2)\} \tag{C.2}$$

is the covariance function of $\eta(\underline{x},t)$. Recall from (76) that we have, to $O(|\underline{\nabla}h|^2)$

$$\eta(\underline{x},t) = \int_{\underline{k}} |dA^{(0)}|\ \{\cos\ (\chi + \mu) + F\ \sin\ (\chi + \mu)\} \tag{C.3}$$

Hence, it follows from (C.3), (C.2) and the fact that μ's are independent random variables uniformly distributed in $(0,\ 2\pi)$ that

$$K_{\eta\eta}(\underline{x}_1,\ \underline{x}_2,\ t_1,\ t_2) = K_{\eta\eta}(\underline{x}_1,\ \underline{x}_2,\ t_2 - t_1) \tag{C.4}$$

$$= \frac{1}{2}\int_{\underline{k}} |dA_1^{(0)}|\ |dA_2^{(0)}|\ \{\cos\ (\chi_1 - \chi_2) + (F_1 - F_2)\ \sin\ (\chi_1 - \chi_2)\}$$

where

$$|dA_1^{(0)}| = |dA^{(0)}(\underline{k},\underline{x}_1)|$$

$$|dA_2^{(0)}| = |dA^{(0)}(\underline{k},\underline{x}_2)|$$

$$\chi_1 - \chi_2 = -\{\int_{\underline{x}_1}^{\underline{x}_2} \underline{k} \cdot d\underline{x} - \omega(t_2 - t_1)\}$$

$$F_1 - F_2 = F(\underline{k},\underline{x}_1,0) - F(\underline{k},\underline{x}_2,0) \qquad \text{(C.5 a,b,c,d)}$$

By definition

$$M_\eta\{\tau(\underline{x},t)\} = E\{\exp\ [i \int_{\underline{X}}\int_{T} \eta(\underline{x},t)\ \tau(\underline{x},t)\ d\underline{x}\ dt\} \qquad \text{(C.6)}$$

Now, we approximate (C.3) by a Riemann sum:

$$\eta(\underline{x},t) = \sum_{n,m} |dA_{nm}^{(0)}|\{\cos\ (\chi_{nm} + \mu_{nm}) + F_{nm}\sin\ (\chi_{nm} + \mu_{nm})\} \qquad \text{(C.7)}$$

where

$$|dA_{nm}^{(0)}| = [\psi^{(0)}(\underline{k}_{nm},\ \underline{x})\ \Delta\underline{k}_{nm}]^{1/2}$$

$$\chi_{nm} = \int^{\underline{x}} \underline{k}_{nm} \cdot d\underline{x} - \omega_{nm} t$$

$$F_{nm} = F(\underline{k}_{nm},\ \underline{x},\ 0)$$

$$\mu_{nm} = \mu(\underline{k}_{nm},\ \underline{x}) \qquad \text{(C.8 a,b,c,d)}$$

From (C.6), (C.7) and the fact that μ_{nm} are uniformly distributed in $(0,2\pi)$, it is immediate that

$$M_\eta\{\tau(\underline{x},t)\} = \prod_{n,m} Z_{nm} \tag{C.9}$$

where

$$Z_{nm} = \frac{1}{2\pi}\int_0^{2\pi} \exp\{i(a_{nm}\cos\mu_{nm} - b_{nm}\sin\mu_{nm})\}\, d\mu_{nm}$$

$$a_{nm} = \int_{\underline{X}}\int_T |dA_{nm}^{(0)}|\, \{\cos\chi_{nm} + F_{nm}\sin\chi_{nm}\}\, \tau(\underline{x},t)\, d\underline{x}\, dt$$

$$b_{nm} = \int_{\underline{X}}\int_T |dA_{nm}^{(0)}|\{\sin\chi_{nm} - F_{nm}\cos\chi_{nm}\}\, \tau(\underline{x},t)\, d\underline{x}\, dt \tag{C.10 a,b,c}$$

Equation (C.10 a) can be integrated as follows:

$$Z_{nm} = \frac{1}{2\pi}\int_0^{2\pi} \exp\{i\,\gamma_{nm}\cos(\mu_{nm} - \delta_{nm})\}\, d\mu_{nm} = J_o(\gamma_{nm}) \tag{C.11}$$

where J_o is the zero-order Bessel function and

$$\gamma_{nm} = [a_{nm}^2 + b_{nm}^2]^{1/2}$$

$$\delta_{nm} = \tan^{-1}\left(-\frac{b_{nm}}{a_{nm}}\right) \tag{C.12 a}$$

Therefore, it follows (C.9) and (C.11) that

$$\ln M_\eta\{\tau(\underline{x},t)\} = \sum_{n,m} \ln J_o(\gamma_{nm}) \tag{C.13}$$

Now, if we let $\Delta \underline{k}_{nm} = \max_{\underline{x}} \Delta \underline{k}_{nm}$, which can be made as small as we please by choosing $\Delta \underline{k}_{nm}$ sufficiently small, it becomes evident from (C.8 a) and (C.10 b,c) that a_{nm} and b_{nm} would be $\theta(\Delta \underline{k}_{nm}^{1/2})$ at the most. Therefore, γ_{nm}, which is of the same order $0(\Delta \underline{k}_{nm}^{1/2})$ by virtue of (C.12 a), can be made sufficiently small, enabling one to write

$$J_o(\gamma_{nm}) = 1 - \frac{1}{4}\gamma_{nm}^2 + 0(\gamma_{nm}^4) \tag{C.14}$$

In this manner, we can rewrite (C.13) as

$$\ln M_\eta\{\tau(\underline{x},t)\} = \sum_{n,m} \ln\{1 - \frac{1}{4}\gamma_{nm}^2 + 0(\gamma_{nm}^4)\}$$

$$= \sum_{n,m} \{-\frac{1}{4}\gamma_{nm}^2 + 0(\gamma_{nm}^4)\} \tag{C.15}$$

where the expansion

$$\ln\{1 - \frac{1}{4}\gamma_{nm}^2\} = -\frac{1}{4}\gamma_{nm}^2 + 0(\gamma_{nm}^4) \tag{C.16}$$

has been used.

With the substitution of γ_{nm}^2, as defined through (C.12 a), and (C.10 b,c), into (C.15), the log-characteristic function can be written as

$$\ln M_\eta\{\tau\} = -\frac{1}{4} \int_{\underline{X}}\int_{\underline{X}}\int_T\int_T [\sum_{n,m} |dA_1^{(0)}|_{nm} |dA_2^{(0)}|_{nm} \{\cos(\chi_1 - \chi_2)_{nm} + \ldots$$

$$\ldots (F_1 - F_2)_{nm} \sin(\chi_1 - \chi_2)_{nm}\}] \tau(\underline{x}_1, t_1) \tau(\underline{x}_2, t_2)\, d\underline{x}_1\, d\underline{x}_2\, dt_1\, dt_2$$

$$+ \sum_{n,m} O(\Delta \underline{k}_{nm}^2) \qquad \text{(C.17)}$$

Note that as $\max_{n,m} \Delta \underline{k}_{nm} = \max_{\underline{x},n,m} \Delta \underline{k}_{nm} \to 0$, the square-bracketed double sum in (C.17) approaches $2\, K_{\eta\eta}(\underline{x}_1, \underline{x}_2, t_1, t_2)$ and (C.17) becomes identical with (C.1). Therefore, $\eta(\underline{x},t)$ is a Gaussian random process.

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RANDOM WAVE–CURRENT INTERACTIONS IN WATER OF VARYING DEPTH

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Abstract—Refraction of incoherent random gravity waves with currents and bottom topography results in spatial variations in the spectral characteristics of the free surface. Prediction of such variations based on the radiation transfer equation is in a simple analytic form for the case of one dimensional inhomogeneities in currents and topography. This analytic form is examined in terms of two-dimensional wave number- and polar frequency–direction spectra along the associated dynamic and kinematic constraints relevant to wave breaking and reflection. Results are specialized to the simplest case of horizontal shear currents in deep and shallow water with explicit examples to illustrate the relative and combined effects of currents and topography on free surface spectra.

INTRODUCTION

REFRACTION of surface waves interacting with currents and/or underwater topography results in spatial variations in their kinematic and dynamic properties. At present, a well-known wave refraction theory exists to predict such variations for linear monochromatic waves. However, description of the sea surface in terms of monochromatic waves is at best an approximation since most realistic sea states have a more complex, randomly irregular structure. This structure is concisely characterized by a two-dimensional mean square spectral distribution over a wave number space or a polar frequency direction space. Consequently, the application of the concepts developed for monochromatic waves interacting with currents and underwater topography to the ptediction of the spatial transformation of spectral characteristics is of interest in the study of random waves and related phenomena. The earliest effort in this direction is by Longuet-Higgins (1956, 1957), who derived the transformation of two-dimensional wave number spectra by refraction over a general topography. Later, Phillips (1966), Hasselmann (1968) and others generalized Longuet-Higgins' results as a systematic energy balance formulation for the prediction of spatially inhomogeneous spectra, taking into account current interactions, various dissipative and generative effects as well. This formulation has been recently extended to nonlinear random waves by Willebrand (1975). However, explicit solutions and applications of the energy balance or radiation transfer equation have been demonstrated only in a few cases. Karlsson's (1969) numerical predictions of the refractive transformation of polar frequency–direction spectra over topographies with parallel and irregular contours, Krasitskiy's (1974) closed form predictions of a similar nature but restricted to parallel contours in essence constitute further explorations and applications on Longuet-Higgins' (1957) depth–refraction solution. Similarly, by also taking into account an approximate form of bottom friction, Collins (1972) numerically computed spatial variations in spectra due to depth refraction. In the case of unidirectional waves interacting with an opposing or following

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current in deep water, the explicit transformation of one-dimensional frequency spectra was given by Phillips (1966, p. 60). The applications of this to current measurements and wave forces were demonstrated by Huang *et al.* (1972) and Tung and Huang (1973).

The motivation here is to explore some straightforward solutions of the radiation transfer equation for a medium with one-dimensional variations in horizontal current components and topography. Attention is also restricted to linear waves, neglecting various nonlinear mechanisms, dissipation and generation effects. Guided by analogous concepts on monochromatic waves, dynamic and kinematic constraints related to wave breaking and reflectionts are examined. Results are specialized in detail to the simplest case of horizontal shear currents in deep and shallow water, and their salient features are illustrated with explicit examples.

DEFINITIONS AND ANALYSIS

An incoherent random wave field can be represented by

$$\eta\,(\mathbf{x}, t) = \sum_n a_n\,(\mathbf{x}, t) \cos\,(\mathbf{k}_n\,.\,\mathbf{x} - \omega t + \mu_n) \tag{1}$$

where μ_n represents random phases uniformly distributed over $(0, 2\,\pi)$; $\mathbf{k}_n$ and ω_n define the vector wave number and frequency, respectively; and a_n is a Fourier amplitude regarded as a slowly varying function of time, t, and position, $\mathbf{x} = (x, y)$ in a horizontal co-ordinate system fixed at still water level. As the vector wave numbers, $\mathbf{k}_n$, are densely distributed over a $\mathbf{k} = (k_1, k_2)$ plane, the corresponding amplitudes, a_n, tend to become dense and also infinitesimal. Under this condition, the function, ψ, represented by

$$\psi\,(\mathbf{k}; \mathbf{x}, t)\,\mathrm{d}\mathbf{k} \simeq \frac{1}{2}\,a_n^2, \tag{2}$$

with $\mathrm{d}\mathbf{k} = \mathrm{d}k_1\,\mathrm{d}k_2$ as a shorthand notation, is defined as the wave number spectral density, describing the distribution of mean square surface deformation over $\mathbf{k}$-space locally, i.e.

$$< \eta^2\,(\mathbf{x}, \mathrm{t}) > = \frac{1}{2} \sum_n a_n^2 = \int_{\mathbf{k}} \psi\,(\mathbf{k}; \mathbf{x}, t)\,\mathrm{d}\mathbf{k}. \tag{3}$$

This distribution will be obviously different if the definition of wave space is changed. For instance, at any point $\mathbf{x}$ where ω and $\mathbf{k}$ are interrelated by a dispersion relation

$$\omega = \Omega\,(\mathbf{k}, \lambda) \tag{4}$$

in which the properties of the propagation medium are characterized by the function $\lambda\,(\mathbf{x}, \mathrm{t})$, we can write for the same mean square surface deformation

$$< \eta^2\,(\mathbf{x}, t) > = \int_{\omega,\theta} \varphi\,(\omega, \theta; \mathbf{x}, t)\,\omega\,\mathrm{d}\omega\,\mathrm{d}\theta \tag{5}$$

where θ, with $(k_1, k_2) = (k \cos \theta, k \sin \theta)$, is the direction of $\mathbf{k}$ relative to, say, x and the distribution φ represents the so-called directional spectral density. ψ and φ are connected by

$$\psi(\mathbf{k}; \mathbf{x}, t) = \frac{\omega}{k}\frac{\partial \Omega}{\partial k} \varphi(\omega, \theta; \mathbf{x}, t). \tag{6}$$

Consider now a medium with nonuniform still water depth h and moving with velocity $\mathbf{U}$ relative to the fixed $(\mathbf{x}, z)$ co-ordinate system. It is assumed that the temporal and spatial variations in h and $\mathbf{U}$ over any given period and wavelength, i.e. $|\nabla h|/kh$, $(\partial h/\partial t)/\omega h$ etc., are much smaller than unity, and that the continuity equation for the mean flow is satisfied (see, e.g. Bretherton and Garrett, 1969, p. 554). We can write (Phillips, 1966, p. 43)

$$\omega = \Omega(\mathbf{k}, \lambda) = \mathbf{U} \cdot \mathbf{k} + \omega' \tag{7}$$

where

$$\omega' = [gk \tanh kh]^{1/2} \tag{8}$$

represents the frequency relative to a coordinate system moving with the current $\mathbf{U}$. Furthermore, the equations for wave rays are given as (see, e.g. Bretherton and Garrett, 1969; Kenyon, 1971)

$$\frac{d\mathbf{x}}{dt} = \frac{\partial \Omega}{\partial \mathbf{k}} \text{ and } \frac{d\mathbf{k}}{dt} = -\frac{\partial \Omega}{\partial \lambda}\frac{\partial \lambda}{\partial \mathbf{x}} \ . \tag{9}$$

Of the preceding expressions, the first describes a wave ray, i.e., the path traced out by an observer moving with the absolute group velocity

$$\mathbf{C}_G = \frac{\partial \Omega}{\partial \mathbf{k}} = \frac{\partial}{\partial \mathbf{k}}(\mathbf{U} \cdot \mathbf{k}) + \mathbf{C}'_G \tag{10}$$

in which

$$\mathbf{C}'_G = \frac{\partial \omega'}{\partial k}\frac{\mathbf{k}}{k} \tag{11}$$

denotes the group velocity relative to $\mathbf{U}$, and the second of (9) describes the change in the vector wave number, $\mathbf{k}$, along the ray.

Under conservative conditions in which dissipation, generation and wave–wave interactions can be neglected, the energy balance reduces to (see, e.g., Willebrand, 1975, p. 125)

$$\left(\frac{\partial}{\partial t} + \mathbf{C}_G \cdot \nabla_2\right)\left(\frac{\psi}{\omega'}\right) = 0 \tag{12}$$

where $\nabla_2 = (\partial/\partial x, \partial/\partial y)$ represents the horizontal gradient operator.

Clearly (12) describes the propagation of the quantity (ψ, ω'), referred to as *wave action spectral density*, in a general time dependent inhomogeneous medium, and implies that

$$\frac{\psi}{\omega'} = \frac{\omega}{k}\frac{\partial \Omega}{\partial k}\frac{\varphi}{\omega'} = \text{constant} \tag{13}$$

along a wave ray. Therefore, this result together with (9) and the initial values of ψ (or φ), $\mathbf{x}$ and $\mathbf{k}$ at one time are sufficient to determine the spectral density $\psi(\mathbf{k}; \mathbf{x}, t)$ or $\varphi(\omega, \theta; \mathbf{x}, t)$. An obvious alternative that avoids the excessive numerical integration involved in ray tracing in general is to solve (12) by a finite difference approximation, this time, together with the irrotationality condition and the kinematical conservation equation for the vector wave number, $\mathbf{k}$ (see, e.g., Phillips, 1966, p. 43).

In order to explore more explicitly what is involved in (13) in the following, we consider the simple and interesting case of steady state conservative random waves propagating from a spatially homogeneous region such as deep water into an inhomogeneous region with a nonuniform depth profile $h(x)$, and traversing a steady nonuniform current field, $\mathbf{U} = [U(x), V(x), W(x, z)]$, as schematically illustrated in Fig. 1. Under these conditions, the

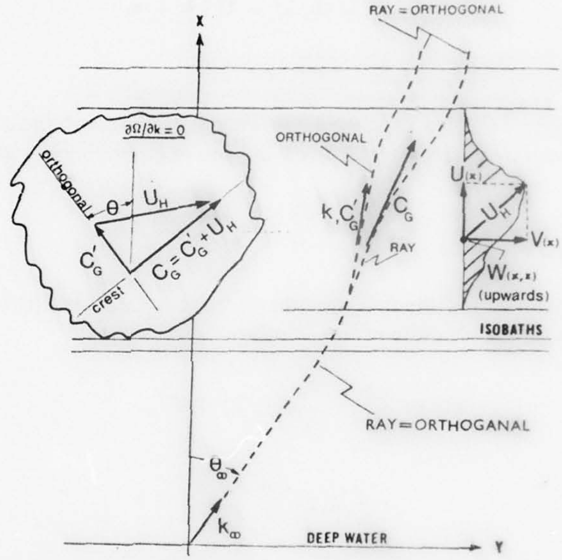

FIG. 1. Schematic diagram showing various definitions and (insert) the limit case $\partial\Omega/\partial k = 0$.

spatial inhomogeneity of the problem is restricted only to the x-direction with

$$k \sin \theta = \text{constant}, \tag{14}$$

and the absolute frequency ω is invariant. The angle, θ, relative to the x-axis is taken to be positive clockwise. It is now immediate, from (7) and (13) with $\mathbf{U}_H = [U(x), V(x), 0]$, that

$$\frac{\psi(\mathbf{k})}{\omega - \mathbf{U}_H \cdot \mathbf{k}} = \frac{\omega}{k}\frac{\partial\Omega}{\partial k}\,\frac{\varphi(\omega, \theta)}{\omega - \mathbf{U}_H \cdot \mathbf{k}} = \text{constant} \tag{15}$$

in the x-direction. It is understood in (15) that ψ and φ are independent of t, and the x-dependency of both quantities are kept implicit for simplicity. Also, to specify various

space-dependent quantities more explicitly, we will, from now on, designate deep water values that are spatially homogeneous in the absence of currents by the subscript (∞), those in finite depth and in the absence of currents by the zero subscript and leave all values in the presence of a current unsubscripted irrespective of any depth consideration. In this manner, it is noted that the simplest form of the constant in (15) is $(\psi/\omega)_\infty$ or $(C_G\, \varphi/k)_\infty$. In terms of these and from (15) it follows, therefore, that the inhomogeneous densities $\psi(\mathbf{k})$ and $\varphi(\omega, \theta)$ are given by

$$\psi(\mathbf{k}) = \left(1 - \frac{\mathbf{U}_H \cdot \mathbf{k}}{\omega}\right) \psi_\infty(\mathbf{k}_\infty) \tag{16}$$

$$\varphi(\omega, \theta) = \frac{k\,(C_G)_\infty \left(1 - \dfrac{\mathbf{U}_H \cdot \mathbf{k}}{\omega}\right)}{k_\infty\,(\partial\Omega/\partial k)}\,\varphi_\infty(\omega, \theta_\infty). \tag{17}$$

The general character of the preceding results indicates that the interaction between a random wave field and nonuniform current–depth effects involves a spatial transformation of the spectral magnitudes, ψ_∞, φ_∞, and their respective wave spaces. However, before we proceed to interpret this transformation, it is appropriate to discuss various kinematical and dynamical constraints embedded in the above equations. First, note from (16) and (17) that the condition

$$\left(1 - \frac{\mathbf{U}_H \cdot \mathbf{k}}{\omega}\right) \geq 0 \tag{18}$$

must be satisfied, in particular, by the components propagating in the direction of the local current, $\mathbf{U}_H$. However, it is also noted that these components must locally have $|\sin\theta| \leq 1$. The latter constraint can be expressed, using (7), (8) and (14), in the form

$$\left(1 - \frac{\mathbf{U}_H \cdot \mathbf{k}}{\omega}\right) \geq [\tanh(kh)\,|\sin\theta_\infty|]^{1/2}. \tag{19}$$

Recognizing that $1 > \tanh(kh)\,|\sin\theta_\infty| > 0$, it becomes evident that the kinematical constraint implied by (19), *only*, is significant. In essence, this constraint is an extension of that given by Longuet-Higgins and Stewart (1961, p. 547) to the inclusion of the effect of nonuniform water depth. At the lower limit where (19) becomes an equality, $|\sin\theta| = 1$, and, therefore, we assume in analogy with monochromatic wave behavior that the associated spectral component is totally reflected (see, e.g., Longuet-Higgins and Stewart, 1961; Kenyon, 1971).

The quantity $\partial\Omega/\partial k$ in the denominator of (17) represents the component of the transport velocity, $\mathbf{C}_G$, along the orthogonal or, equivalently, in the propagation direction ($\mathbf{k}/k$) of a component wavelet. Therefore, locally, the condition

$$\frac{\partial\Omega}{\partial k} = \mathbf{C}_G \cdot \frac{\mathbf{k}}{k} = C'_G + \mathbf{U}_H \cdot \frac{\mathbf{k}}{k} > 0 \tag{20}$$

must be satisfied. In other words, the local group velocity, C'_G, must be opposite in direction and larger in magnitude relative to the horizontal current component, $\mathbf{U}_H \cdot (\mathbf{k}/k)$, in the direction of wave propagation. In the limit condition when $C'_G = -\mathbf{U}_H \cdot (\mathbf{k}/k)$, as schematically illustrated in the insert to Fig. 1, the associated spectral component can no longer propagate against the current in that direction. Theoretically, the local spectral magnitude, φ, becomes infinite. As in the case of monochromatic waves (see, e.g., Longuet-Higgins and Stewart, 1961). this suggests that these spectral components will tend to diminish or attenuate by *wave breaking* and, possibly, by a lateral stretching in the crest direction before this point is reached. Note from (16) that the corresponding spectral magnitude, ψ, over **k**-space remains always bounded (i.e., ≤ 2). It is, however, evident that (16) as well as all other results and definitions of the preceding analysis will lack validity near the critical point.

In summarizing the preceding discussion now, we may conclude that the spatial transformations of the spectral magnitudes, ψ_∞ and φ_∞ via (16) and (17) are subject to the reflection and breaking constraints (19) and (20), respectively. The transformations of the incident densities, φ_∞ and ψ_∞ are, therefore, continuous in x for all but the attenuated and reflected components. The regions of the incident, $\mathbf{k}_\infty$ — or ω, θ_∞-space that violate either of these constraints at a point, x, are cut-off, or simply deleted beyond that point, assuming the absence of any sort of interaction between the attenuated and/or reflected components and those remaining. However, the original incident wave space is to be properly modified, as will be illustrated later with examples, by the reflected components in a manner consistent with the steady state assumption.

In general, the spatial transformation of the magnitude ψ_∞ is entirely due to the current interaction in contrast with the transformation of φ_∞ that involves the combined current-depth effects. The spectral magnitude, ψ, relevant to components with a locally opposing angle to the current (i.e. $\mathbf{U}_H \cdot \mathbf{k} < 0$) are amplified in comparison with ψ_∞, and those propagating with the current (i.e., $\mathbf{U}_H \cdot \mathbf{k} > 0$) are suppressed irrespective of any depth consideration. In the case of φ, the current effect on components locally opposing the current is qualitatively the same as in the case of ψ, i.e., amplification. Howevre, for components propagating with the current, the net effect of the combined current–depth interactions could be an amplification or suppression of φ relative to φ_∞ depending, from (17), on whether the associated ω, θ_∞ values satisfy the condition

$$k_\infty \frac{\partial \Omega}{\partial k} < k(C_G)_\infty \left(1 - \frac{\mathbf{U}_H \cdot \mathbf{k}}{\omega}\right) \tag{21}$$

or not, respectively.

The local **k**-space associated with the transformed magnitude ψ is distorted by the combined current–depth effects in terms of the magnitude k as well as the direction θ of the wave number vector **k**. On the other hand the distortion in the polar ω, θ-space is entirely due to the spatial dependence of θ. Since the complete spectral transformations require the mapping of the density ψ or φ desirably in the form of contours in the local wave space, the invariant nature of the frequency ω makes the polar ω, θ-space particularly advantageous to work with. Therefore, the discussion will be restricted to this space from now on.

HORIZONTAL SHEAR CURRENT CASE, EXAMPLES, AND VARIATION OF MEAN ENERGY

Considering the simplest case of a horizontal shear current, $\mathbf{U}_H = [0, V(x), 0]$, and using (7), (8), (10), (11) and (14) result in the following:

$$\omega^2\left[1 - \frac{V(x)}{C_\infty}\sin\theta_\infty\right]^2 = gk\tanh kh \tag{22}$$

represents the dispersion relation, and

$$C' = C_\infty\left[1 - \frac{V(x)}{C_\infty}\sin\theta_\infty\right]^{-1}\tanh kh \tag{23}$$

is the phase speed relative to the current. Also

$$\begin{aligned}\sin\theta &= (k_\infty/k)\sin\theta_\infty \\ &= (\tanh kh\left[1 - \frac{V(x)}{C_\infty}\sin\theta_\infty\right]^{-2}\sin\theta_\infty\end{aligned} \tag{24}$$

$$k = k_\infty\left[1 - \frac{V(x)}{C_\infty}\sin\theta_\infty\right]^2(\tanh kh)^{-1} \tag{25}$$

$$\frac{\partial\Omega}{\partial k} = \frac{\omega}{k}\left[n + (1-n)\frac{V(x)}{C_\infty}\sin\theta_\infty\right] \tag{26}$$

where

$$n = \frac{C'_G}{C'} = \frac{1}{2}\left[1 + \frac{2kh}{\sinh 2kh}\right]\ . \tag{27}$$

On substitution from these (13), (19) and (20) become, respectively,

$$\varphi(\omega,\theta) = \frac{k\,C_\infty\left[1 - \frac{V(x)}{C_\infty}\sin\theta_\infty\right]}{2\,k\left[n + (1-n)\frac{V(x)}{C_\infty}\sin\theta_\infty\right]}\varphi_\infty(\omega,\theta_\infty) \tag{28}$$

$$\left[1 - \frac{V(x)}{C_\infty}\sin\theta_\infty\right] \geq [(\tanh kh)\sin\theta_\infty]^{1/2},\ \theta_\infty > 0 \tag{29}$$

$$\frac{n}{(1-n)}C_\infty > -V(x)\sin\theta_\infty,\ \theta_\infty < 0. \tag{30}$$

At this point, it is worthwhile to discuss some examples illustrating what is involved in the preceding transformations and constraints. For simplicity in presentation the examples

will be confined to deep and shallow water conditions where various key definitions involved in the transformation (28) become analytically tractable. Furthermore, it will be convenient to nondimensionalize various quantities in terms of the value $V_m = \max V(x)$ as follows:

$$\omega^* = \frac{\omega V_m}{g}, \; k^* = \frac{k V_m^2}{g}, \; h^* = \frac{gh(x)}{V_m^2}, \; \varphi^* = \left(\frac{V_m}{g}\right)^2 \varphi. \tag{31}$$

The explicit form of the incident spatially homogeneous deep water spectral density in both examples is assumed to be

$$\varphi^*_\infty(\omega^*, \theta_\infty) = \begin{cases} 1 \; ; \; 1/2 \leq \omega^* \leq 3, \; |\theta_\infty| \leq \pi/2. \\ 0 \; ; \; \text{otherwise} \end{cases} \tag{32}$$

The incident ω^*, θ_∞-space has, therefore, a semi-annular shape as schematically illustrated in Figs. 2 and 7 (continuous line boundary). A spectral density of the form (32) by no means

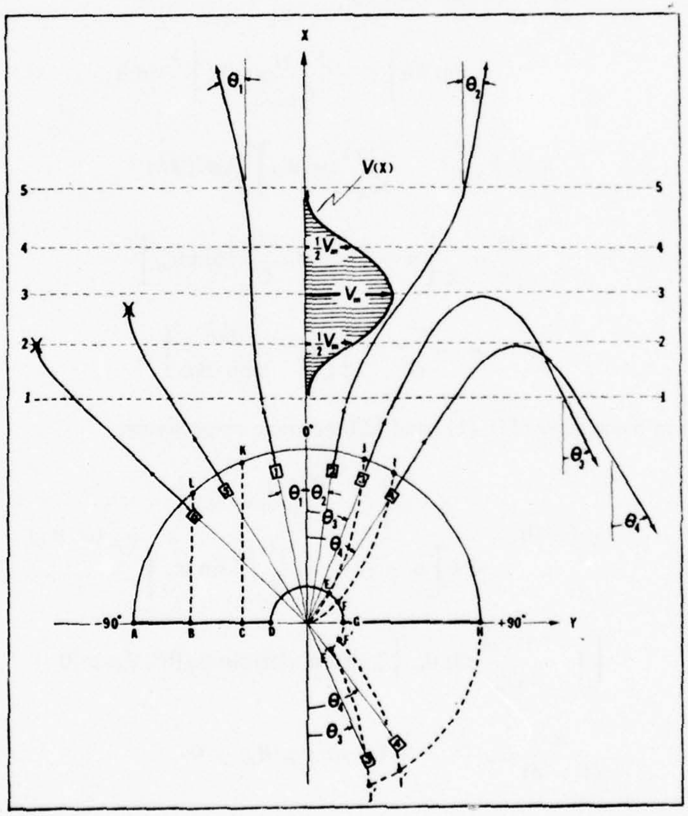

FIG. 2. Example 1: Current interactions in deep water. Definition sketch showing the qualitative effect of the shearing current on wave orthogonals for various angles of entry θ_∞.

characterizes any oceanic situation. This artificial form is of convenience here in that the deformation of the incident ω, θ_∞-space and contours of density amplification or suppression can be followed easily. Also, because of the unity magnitude, the transformation of a more realistic spectral density is readily obtained with a simple multiplication of amplitudes over regions of ω, θ_∞-space coincident with the one chosen here.

Example 1: *Current interactions in deep water*

In deep water (24), (29), (30) and (28) become, respectively,

$$\sin\theta = \left[1 - \frac{V(x)}{V_m}\,\omega^*\sin\theta_\infty\right]^{-2}\sin\theta_\infty \tag{33}$$

$$\frac{V(x)}{V_m}\,\omega^*\sin\theta_\infty \le (1 - \sin^{1/2}\theta_\infty),\quad \theta_\infty > 0 \tag{34}$$

$$-\frac{V(x)}{V_m}\,\omega^*\sin\theta_\infty < 1,\ \theta_\infty < 0 \tag{35}$$

$$\varphi^*(\omega^*, \theta)/\varphi^*_\infty(\omega^*, \theta_\infty) = A \tag{36}$$

where the *spectral amplification factor*, A is in the form

$$A = \left[1 - \frac{V(x)}{V_m}\,\omega^*\sin\theta_\infty\right]^5\left[1 + \frac{V(x)}{V_m}\,\omega^*\sin\theta_\infty\right]^{-1}. \tag{37}$$

The shear current field, $V(x)$, shown schematically in Fig. 2, consists of regions of monotonic increase and decrease. It is evident from (37) and (33) tha

$$A\begin{cases} > 1\ (\theta_\infty < 0) \\ = 1\ (\theta_\infty = 0) \\ < 1\ (\theta_\infty > 0) \end{cases} \tag{38}$$

and the contours of density $\varphi^* = A =$ constant, corresponding to various fixed values of the product $\omega^*\sin\theta_\infty$, are straight lines parallel to the direction of normal incidence ($\theta_\infty = \theta = 0°$) over both ω^*, θ_∞- and ω^*, θ-space. The breaking constraint (35) indicates the regions of the incident ω^*, θ_∞-space to be deleted in a progressive manner dependent on the ratio $V(x)/V_m$ and whether $\mathrm{d}V/\mathrm{d}x > 0$ or $\mathrm{d}V/\mathrm{d}x < 0$. For instance, referring to Fig. 2, the spectral components in the region ABL of the incident ω^*, θ_∞-space are entirely eliminated by the time waves propagate to the section (2–2) where $V(x) = 0.5\ V_m$. Similarly, the components in the region BCKL gradually dissipate as waves advance from (2–2) to (3–3). However, beyond (3–3) where $\mathrm{d}V/\mathrm{d}x < 0$, wave breaking has no influence on the spectral components to the right of the line CK. For $\theta_\infty > 0$, the reflection constraint (34) suggests that, as waves propagate from (1–1) to (2–2), the part IFGH of the incident ω^*,

θ_∞-space be eliminated at (2–2). Similarly, from (2–2) to (3–3), the components in the region EFIJ are reflected leaving only the part CDEJK of the incident wave space beyond (3–3) where breaking and reflection become immaterial. However, for a steady state, the reflected spectral components must be properly accounted for since they propagate back in the direction $(\pi - \theta)$ locally and modify the incident wave space. Therefore, the deep water density $\varphi_\infty^*(=1)$ and the associated ω^*, θ_∞-space must be modified as shown in Fig. 2 by dash lines with the region Ge′J′H representing the mirror image of the part GEJH that has been progressively reflected back as waves propagated from (1–1) to (3–3).

Figures 3–5 correspond to a simple mapping of the transformed densities over their local ω^*, θ-spaces at any point along the lines (2–2), (3–3) and (4–4) of Fig. 2, respectively. By the lettering in Fig. 2 and the corresponding ones in these figures, the local distortion of the various regions of the incident wave space, and the amplification or suppression of the spectral magnitudes are easily observed. It is noted in each case that the spectral amplitude of components with normal incidence to the current $(\theta_\infty = 0°)$ traverse the current unaffected, whereas those with opposing (following) angles of entry are amplified (suppressed). In Fig. 3, the region FE′J′I represents those components reflected back in between (2–2) and (3–3). Finally, Fig. 6 illustrates the spectral density and the associated wave space at any point on and beyond the line (5–5) where $V(x) = 0$. The spectral magnitudes $(= 1)$ and the wave space in this figure are, therefore, exactly the same as the CDEJK part of Fig. 2.

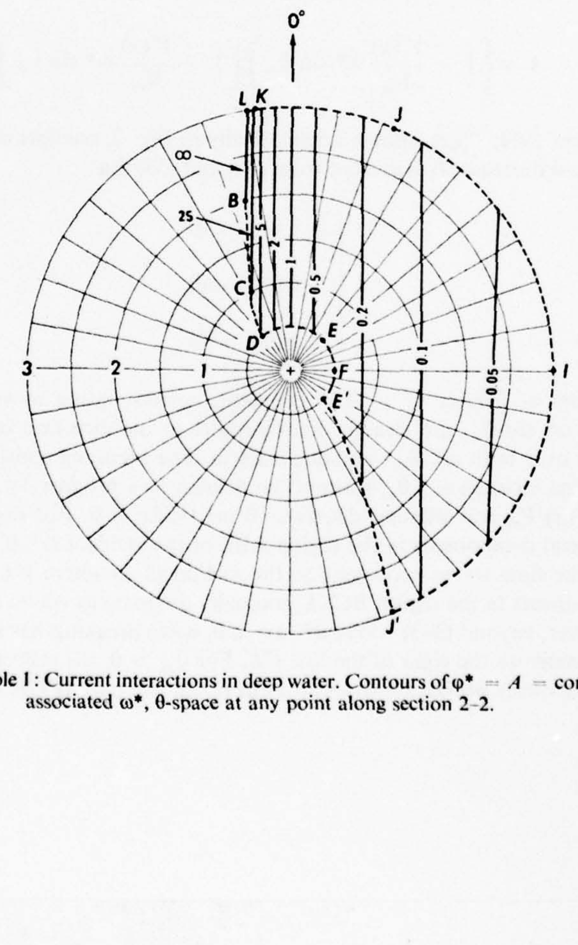

FIG. 3. Example 1: Current interactions in deep water. Contours of $\varphi^* = A =$ const. and the associated ω^*, θ-space at any point along section 2–2.

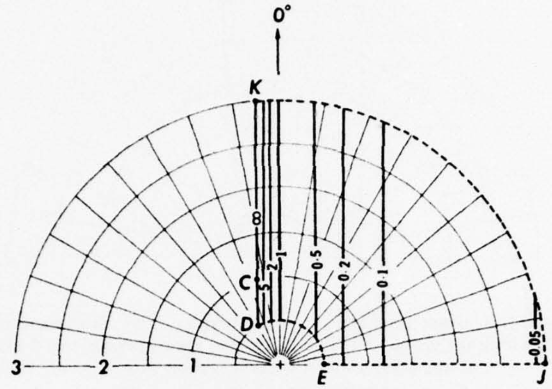

FIG. 4. Example 1: Current interactions in deep water. Contours of $\varphi^* = A =$ const. and the associated ω^*, θ-space at any point along section 3–3.

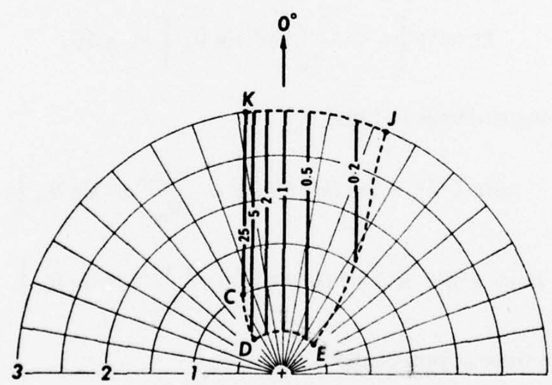

FIG. 5. Example 1: Current intereactions in deep water. Contours of $\varphi^* = A =$ const. and the associated ω^*, θ-space at any point along section 4–4.

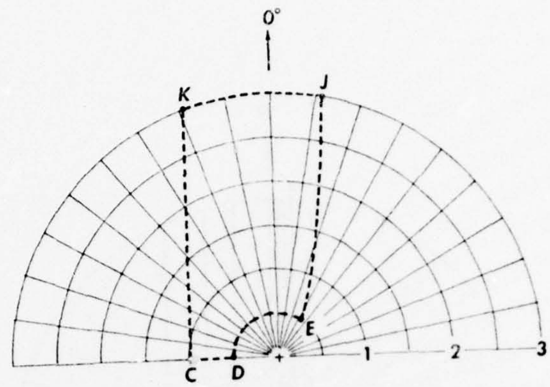

FIG. 6. Example 1: Current interactions in deep water. The directional spectral density $\varphi^* = \varphi_\infty^*$ ($= 1$, within the dashed-line boundary) and the associated ω^*, θ-space ($= \omega^*, \theta_\infty$) at any point on and beyond section 5–5.

Example 2: Current interactions in shallow water

By virtue of the definitions (31), we may rewrite the dispersion relation (22) as:

$$\tanh k^*h^* = \frac{(\omega^*)^2\left[1 - \frac{V(x)}{V_m}\,\omega^* \sin\theta_\infty\right]^2}{k^*} \qquad (39)$$

Under the shallow water condition

$$h^*(\omega^*)^2\left[1 - \frac{V(x)}{V_m}\,\omega^* \sin\theta_\infty\right] \leq \pi/10 \qquad (40)$$

we find [see, e.g., Longuet-Higgins, 1956]

$$\tanh k^*h^* \simeq \omega^*(h^*)^{1/2}\left[1 - \frac{V(x)}{V_m}\,\omega^* \sin\theta_\infty\right] \qquad (41)$$

and

$$\frac{\partial}{\partial k^*}(\tanh k^*h^*) \simeq 2\,\omega^*(h^*)^{1/2}\left[1 - \frac{V(x)}{V_m}\,\omega^* \sin\theta_\infty\right]. \qquad (42)$$

It is immediate from these approximations that

$$C' \simeq C'_G \simeq V_m(h^*)^{1/2} \qquad (43)$$

and (24), (29), (30) and (28) become, respectively,

$$\sin\theta \simeq \omega^*(h^*)^{1/2}\left[1 - \frac{V(x)}{V_m}\,\omega^* \sin\theta_\infty\right]^{-1} \sin\theta_\infty \qquad (44)$$

$$\frac{V(x)}{V_m}\,\omega^* \sin\theta_\infty \leq [1 + (h^*)^{1/2}]^{-1},\ \theta_\infty > 0 \tag{45}$$

$$-\frac{V(x)}{V_m}\,\omega^* \sin\theta_\infty < \frac{n}{l-n} \simeq \infty,\ \theta_\infty < 0 \tag{46}$$

$$A = \varphi^*(\omega^*, \theta)/\varphi_\infty^*(\omega^*, \theta_\infty) \simeq \left[1 - \frac{V(x)}{V_m}\,\omega^* \sin\theta_\infty\right]^3 [2h^*(\omega)^{*2}]^{-1}. \tag{47}$$

As a specific example here, consider a current of the form schematically illustrated in Fig. 7, together with the shallow water depth profile. We see that the breaking constraint

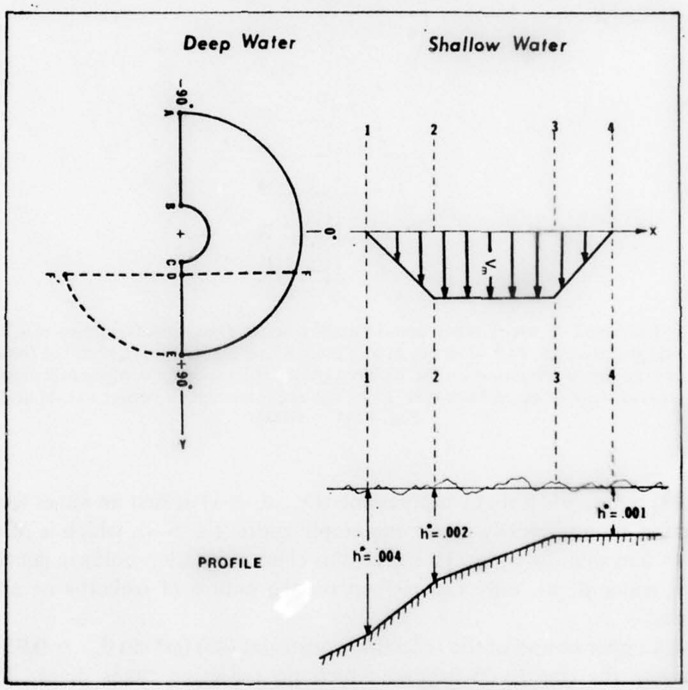

FIG. 7. Example 2: Current interactions in shallow water. Definition sketch in plan view (top) and profile.

(46) is redundant in this case. That is to say, no component with finite ω^* in the local ω^*, θ_0-space just before the current (see Figs. 7 and 8) will ever attain a breaking condition on account of excessive depth refraction which distorts and focuses the wave space in the direction of normal incidence to the current. Note, from (15) and (17) with $\mathbf{U}_H = 0$, that the spatial modification of spectral amplitudes due to depth effects is described by

$$A = \frac{\varphi^*}{\varphi} = \frac{(C_G/k)_\infty}{(C_G/k)}\,. \tag{48}$$

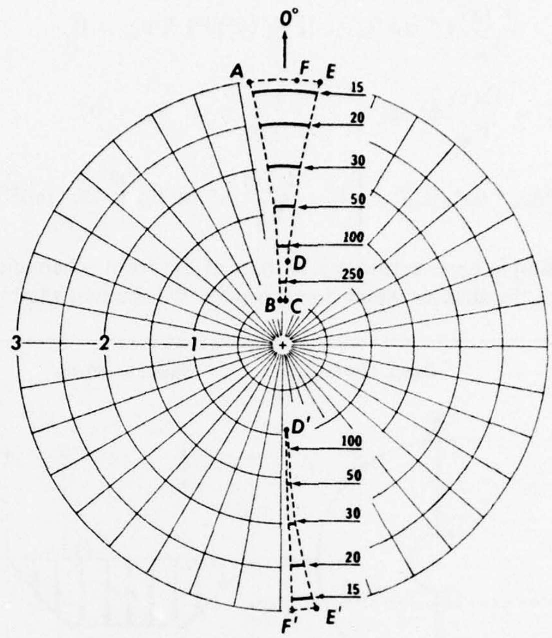

FIG. 8. Example 2: Current interactions in shallow water. Diagram of contours of $\varphi_0^* = A =$ const. and the associated ω^*, θ_0-space at any point along the line (1–1), showing the dramatic effect of pure depth refraction on the incident (ABCDEF) spectral components as well as on those progressively reflected back (DF′E) by the shear current in between (1–1) and (2–2) of Fig. 7 ($h^* = 0.004$).

Evidently (48) represents a slight suppression (i.e., $A < 1$) at first as waves feel the bottom, then becomes a monotonically increasing amplification ($A > 1$), which is of the form (47) with $V(x) = 0$ in shallow water. However, this characterization holds in general depending on the local water depth only rather than on the nature of isobaths or any directional consideration.

The lowest upper bound of the reflection constraint (45) ($\omega^* \sin \theta_\infty \simeq 0.9572$) is realized at points along the line (2–2) beyond which no reflection takes place. This bound is schematically illustrated with the dash line DF in Fig, 7. Therefore, as waves reach (2–2), the components in the DFE region of ω^*, θ_∞-space are progressively reflected, propagating back and locally modifying the incident wave field as in the previous example. Correspondingly, the steady state deep water wave space would be as shown in the same figure, with the region DF′E bounded by dash lines representing the mirror image of DFE. Presented in the subsequent Figs. 8–11 are the local spectral contours and wave spaces corresponding to any point (1–1), (2–2), (3–3) and (4–4) of Fig. 7, respectively. In particular, Fig. 8 illustrates the depth affects discussed in the preceding on the incident and reflected spectral components just before entry to the shear current. Figures 9 and 10 present the combined current–depth effects. Qualitatively, components with an opposing angle of entry ($\theta_\infty < 0$) are further amplified and focused in the direction of normal incidence. Obviously, for components

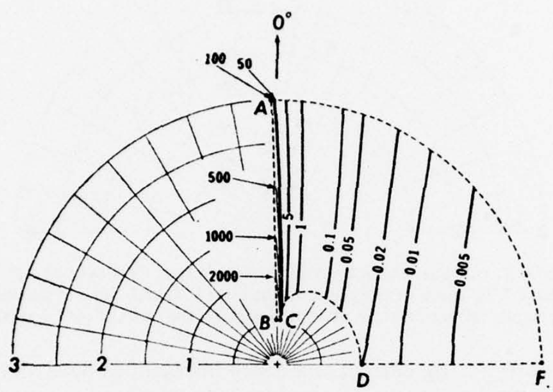

FIG. 9. Example 2: Current interactions in shallow water. Contours of $\varphi^* = A =$ const. and the associated ω^*, θ-space at any point on the line (2–2), showing the combined effects of depth and current interactions ($h^* = 0.002$).

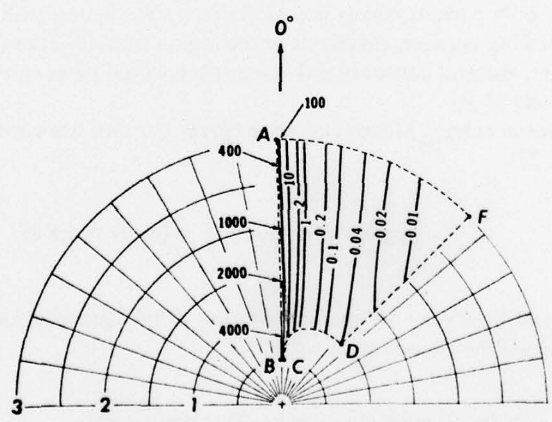

FIG. 10. Example 2: Current interactions in shallow water. Contours of $\varphi^* = A =$ const. and the associated ω^*, θ-space at any point on the line (3–3), showing again the combined effects of depth and current interactions ($h^* = 0.001$).

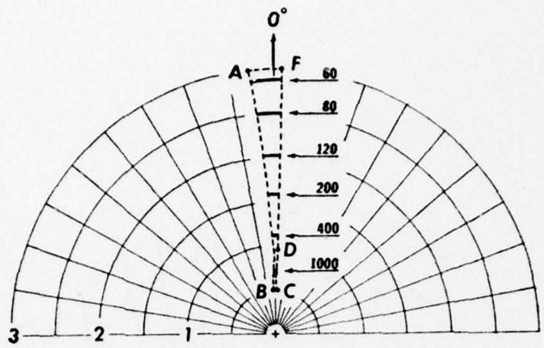

FIG. 11. Example 2: Current interactions in shallow water. Contours of $\varphi_0^* = A =$ const. and the associated ω^*, θ_0-space at any point on the line (4–4) and, beyond showing the effect of pure depth refraction right after waves cross the current ($h^* = 0.001$).

following the current ($\theta_\infty > 0$), current and depth interactions have an opposing influence and their net effect is in general characterized by (47). Therefore, local spectral amplitudes for which ω^*, θ_∞ values satisfy (violate) the condition $2h\,(\omega^*)^2 < [1 - (V/V_m)\,\omega^* \sin\theta_\infty]^3$ remain amplified (suppressed) relative to the corresponding amplitudes in deep water as illustrated in Fig. [illegible] to the left (right) of unity contour line. Components dominated over by the current (i.e., $A < 1$) are in effect swept along with the current as they propagate from (1–1) to (2–2). Between (2–2) and (3–3) where the current is uniform, only depth effects are operative in amplifying and focusing spectral components further relative to those at (2–2). This is illustrated for any point along (3–3) in Fig. 10. Finally, beyond (3–3) where the depth is uniform, only current effects are of concern, interacting with the wave field in a reverse and diminishing manner, exactly as in the region from (3–3) to (5–5) of the previous example. Therefore, spectral contours and wave space would be as shown in Fig. 11 at any point on and beyond (4–4).

Comments on mean energy. Mean total wave energy per unit horizontal area is defined by (Phillips, 1966, p. 27)

$$E = \rho g < \eta^2 > = \rho g \int_{\omega,\theta} \varphi(\omega, \theta)\, \omega\, d\omega\, d\theta. \tag{49}$$

In order to comment briefly on the spatial variation of this quantity for the particular case of interest, here, it will be expedient to write

$$\varphi(\omega, \theta)\, \omega\, d\omega\, d\theta = \varphi(\omega, \theta)\, \omega\, d\omega \frac{\partial\theta}{\partial\theta_\infty}\, d\theta_\infty \tag{50}$$

where, by virtue of (24),

$$\frac{\partial\theta}{\partial\theta_\infty} = \frac{k_\infty}{k \cos\theta}\left(\cos\theta_\infty - \frac{\sin\theta_\infty}{k}\frac{\partial k}{\partial\theta_\infty}\right) \tag{51}$$

and, from (25),

$$\frac{\partial k}{\partial \theta_\infty} = -\frac{k\,k_\infty\,V(x)\cos\theta_\infty}{n\,\omega\left[1 - \dfrac{V(x)}{C_\infty}\sin\theta_\infty\right]} \quad . \tag{52}$$

On substitution from (50), (51) and (52), (49) can be written as

$$E = \rho g \int\limits_{\omega,\,\theta_\infty\,\varepsilon\,R} \frac{(n \sin 2\theta)_\infty}{n \sin 2\theta}\,\varphi_\infty(\omega, \theta_\infty)\,\omega\,d\omega\,d\theta_\infty \tag{53}$$

where R represents the region of the incident ω, θ_∞-space that is not deleted by wave breaking and/or reflection as waves propagate to the point of interest. This expression is a generalization of various well-known results associated with monochromatic waves and, of course, with a narrow-band random wave field. For instance, in the case of monochromatic waves with the incident height, H_∞, frequency ω, direction θ_∞, and energy $E_\infty = (1/8)\rho g\,H^2{}_\infty$, one can show from (53) that

$$E = \frac{(n \sin 2\theta)_\infty}{n \sin 2\theta}\,E_\infty \quad . \tag{54}$$

This obviously reduces to the familiar form

$$E_0 = \frac{(C_G \cos\theta)_\infty}{(C_G \cos\theta)_0}\,E_\infty \tag{55}$$

in the absence of any currents (see, e.g., Phillips, 1966, p. 53), and to

$$E = \frac{\sin 2\theta_\infty}{\sin 2\theta}\,E_\infty \tag{56}$$

in deep water with the presence of currents (Longuet-Higgins and Stewart, 1961, p. 547).

CONCLUDING REMARKS

A mean-square spectral distribution constitutes a concise characterization for incoherent random gravity waves. In the preceding, the spatial transformation of this distribution by refraction due to currents and bottom topography has been presented in closed form and with explicit examples in the particular case of one-dimensional variations in currents and topography. Clearly, the omission of various other realistic effects such as nonlinear mechanisms, frictional dissipation, generation, etc., requires a tentative judgment on the results here as in the case of linear monochromatic waves. It is seen, however, that currents and underwater topography can have a dramatic influence on the spectral characteristics of random waves. Shear currents, in effect, act as a filter, dissipating and/or reflecting certain components while transmitting others with a substantial amplification or reduction in amplitude. These interactions are, then, modified further and become more complex in shallower depths with depth refraction and shoaling. Various aspects of these interactions,

particularly kinematics, are well known in the case of deterministic monochromatic waves. The primary motivation in the extension of these concepts to incoherent random waves lies in the more general character of results in terms of a spatially inhomogeneous spectral distribution and, therefore, in the fact that it supplements the statistical approach to the description of the real ocean surface. This approach is presently considered by many as a more fruitful one than a deterministic theory based on a monochromatic wave train.

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13. ABSTRACT

The report consists of three parts. The first part presents a new method of determination of the change of random wave energy as the waves propagate from offshore towards the shore. The second part presents a new random wave model taking into account the varying water depth in shallow water. The third part presents a method of calculating the two dimensional random wave field near shore in terms of directional spectrum, taking into account the wave-current interactions.

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